# Math 310
# September 21, 2020 Lecture

Steven G. Krantz

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Figure: This is your instructor.

Let us look back at the ideas of this chapter and comment on the difference between truth and provability.

An elementary statement such as

$$A = \text{ “George is tall.”}$$

has a truth value assigned to it. It is either true or false. From the point of view of mathematics, there is nothing to prove about this statement. Likewise for the statement

$$B = \text{ “Barbara is wise.”}$$

On the other hand, the statement

$$A \vee B.$$

is subject to mathematical analysis. Namely, it is true if at least one of A or B is true. Otherwise it is false.

Any statement that is true regardless of the truth value of its individual components is called a *tautology*. An example of a tautology is

$$B \Rightarrow (A \vee \sim A).$$

This statement is true all the time—regardless of the truth values of A and B. Set up a truth table to satisfy yourself that this is the case.

Another example of a tautology is

$$(A \Rightarrow B) \Leftrightarrow (\sim A \vee B).$$

Again, you may verify that this is a tautology by setting up a truth table.

So we have two ways to think about whether a certain statement is valid all the time: (i) to substitute in all possible truth values, and (ii) to prove the statement from elementary first principles. We have seen two examples of (i). Now let us think about method (ii).

In order to provide an example of a provable statement, we must isolate in advance what are the syllogisms that we assume in advance to be true, and what rules of logic are allowed. In a formal treatment of logic, such as [SUP] or [STO], we would begin on page 1 of the book with these syllogisms and rules of logic and then proceed rigidly, step by step. At each stage, we would have to check which rule or syllogism is being applied. The present book is *not* a formal treatment of logic. It is in fact a more intuitive approach. For the remainder of the section, however, we lapse into the formal mode so that we may learn more carefully to distinguish truth from provability.

First, which rules of logic do we allow? There is only one: *modus ponendo ponens* is the only rule of logic (this is the rule that $A \Rightarrow B$ together with $A$ entails $B$). Now the other assumptions are these: for the present discussion we take $\sim$ and $\vee$ as our only primitive connectives. Then

N1 $A \Rightarrow B$ is an abbreviation for $\sim A \vee B$.

N2 $A \wedge B$ is an abbreviation for $\sim (\sim A \vee \sim B)$.

Axiom 1 $(C \vee C) \Rightarrow C$

Axiom 2 $C \Rightarrow (C \vee B)$

Axiom 3 $(C \vee B) \Rightarrow (B \vee C)$

Axiom 4 $(B \Rightarrow A) \Rightarrow \big([C \vee B] \Rightarrow [C \vee A]\big)$

Notice that Axioms 1–4 are all “intuitively obvious.” Any good axiom should have this feature, because we do not verify or prove axioms. The axioms are our starting place; nothing comes before the axioms. We just accept them. For example, let us think about Axiom 2: If we assume that C is true, then it is certainly the case that $C \vee B$ is true. In this way, we satisfy our intuition that Axiom 2 is a reasonable axiom. You may check the other axioms for yourself using similar reasoning.

In some more formal treatments, additional rules of logic are enunciated. The Axiom of Substitution (Axiom Schema of Replacement) is also an important rule of logical reasoning. We shall say more about it later.

In a formal treatment of proof theory (see [BUS, p. 5 ff.]), we sometimes specify—in addition to *modus ponens*—a system of logical axioms that allow the inference of "self-evident" tautologies from no hypotheses. One such system is this (see [BUS]):

(i) $p \Rightarrow (q \Rightarrow p)$

(ii) $(p \Rightarrow q) \Rightarrow [(p \Rightarrow \sim q) \Rightarrow \sim p]$

(iii) $(p \Rightarrow q) \Rightarrow [(p \Rightarrow (q \Rightarrow r)) \Rightarrow (p \Rightarrow r)]$

(iv) $(\sim\sim p) \Rightarrow p$

(v) $p \Rightarrow (p \vee q)$

(vi) $(p \wedge q) \Rightarrow p$

(vii) $q \Rightarrow (p \vee q)$

(viii) $(p \wedge q) \Rightarrow q$

(ix) $(p \Rightarrow r) \Rightarrow [(q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r)]$

(x) $p \Rightarrow [q \Rightarrow (p \wedge q)]$

together with two axiom schemes for the quantifiers:

(xi) $\mathrm{A}(t) \Rightarrow \exists x, \mathrm{A}(x)$

(xii) $\forall x, \mathrm{A}(x) \Rightarrow \mathrm{A}(t)$

and two quantifier rules of inference:

(xiii) $[C \Rightarrow \mathrm{A}(x)] \Rightarrow [C \Rightarrow \forall x, \mathrm{A}(x)]$

(xiv) $[\mathrm{A}(x) \Rightarrow C] \Rightarrow [\exists x, \mathrm{A}(x) \Rightarrow C]$

We refer to axioms (i)–(x) as $\mathcal{F}$, in honor of Gottlob Frege (1848–1925). It is a remarkable fact that $\mathcal{F}$ is complete in the sense that any tautological statement of the propositional calculus can be proved using $\mathcal{F}$.

For the purposes of the present book, *modus ponendo ponens* will be the primary rule of reasoning. The reader can safely worry about no others. Any assertion that we assert to be *provable* must be derivable, using the logical rule *modus ponendo ponens*, from our notational conventions and these axioms. As an illustration, let us prove the statement $\sim (B \wedge \sim B)$. [Note that you can easily check this with a truth table; so it *is* a tautology. But now we want to *prove* it from (i) our *definitions*, (ii) our *axioms*, and (iii) our *rules of logic*.]

Now N2 above shows that the statement that we wish to prove is just $\sim B \vee B$. [We have used here the logical equivalence of $\sim\sim B$ and $B$. The details of this equivalence are left to you.] It is more natural to prove *this* statement since our axioms are formulated in terms of the connective $\vee$. Here is our proof:

| | | |
|---|---|---|
| (1) | $(B \vee B) \Rightarrow B$ | by Axiom 1 |
| (2) | $[(B \vee B) \Rightarrow B] \Rightarrow$ $([\sim B \vee [B \vee B]] \Rightarrow [\sim B \vee B])$ | by Axiom 4 |

| | | |
|---|---|---|
| (3) | $([\sim B \vee [B \vee B]] \Rightarrow [\sim B \vee B])$ | by modus ponendo ponens applied to (1), (2) |
| (4) | $(B \Rightarrow (B \vee B)) \Rightarrow (B \Rightarrow B)$ | applying N1 to (3) |
| (5) | $B \Rightarrow (B \vee B)$ | by Axiom 2 |
| (6) | $B \Rightarrow B$ | by modus ponendo ponens applied to (4), (5) |
| (7) | $\sim B \vee B$ | applying N1 to (6) |

That completes the proof.

Implicit in this last discussion is the question of why we can restrict attention to just the connectives $\sim$ and $\vee$. In fact, all the other connectives can be expressed in terms of just these two. As an instance, $A \wedge B$ is logically equivalent to $\sim(\sim A \vee \sim B)$. Likewise, $A \Rightarrow B$ is logically equivalent with $(\sim A) \vee B$. These statements can be checked with truth tables. It can also be shown that $\sim$ and $\wedge$ can be used to generate all the other connectives. Some combinations are not possible: $\vee$ and $\wedge$ *cannot* be used to form a statement that is equivalent with $\sim$. Again, you can use truth tables to confirm this assertion.

It is natural to ask, and we raised this question implicitly in an earlier lecture, whether every tautology is provable (that every provable statement is a tautology is an elementary corollary of our logical structure, or see [STO, p. 152]). That this is so is Frege's theorem. This statement is summarized by saying that elementary sentential logic is *complete*.

In fact Gödel (1906–1978) proved in 1930 that the so-called first-order predicate calculus is complete. The first-order predicate calculus is essentially the logic that we have described in the present chapter: it includes elementary connectives, the quantifiers $\forall$ and $\exists$, and statements $P$ with one or more (but finitely many) variables $x_1, \ldots, x_k$. Thus, according to Gödel, any provable statement in this logic is true and, more profoundly, any true statement is provable. Gödel went on to construct a model for any consistent system of axioms. Interestingly, his proof requires the Axiom of Choice.

Gödel's more spectacular contribution to modern thought is that, in any logic that is complex enough to contain arithmetic, there are sensible statements that cannot be proved either true or false. More precisely, there are true statements that cannot be proved; and there are false statements that cannot be disproved. For example, Peano's arithmetic contains statements that cannot be proved either true or false. A rigorous discussion of this celebrated "incompleteness theorem" is beyond the scope of the present book. Suffice it to say that Gödel's proof consists of making an (infinite) list of all provable statements, enumerating with a system of "Gödel numbers", and then constructing a new statement that differs from each of these. Since the constructed statement could not be on the list, it also cannot be provable. For further discussion of Gödel's ideas, see [DAV], [NAN], [SMU].

Theoretical computer scientists have shown considerable interest in the incompleteness theorem. For a computer language—even an expert system—can be thought of as a logical theory. Gödel's theorem says, in effect, that there will be statements formulable in any sufficiently complex language that cannot be established through a sequence of logical steps from first principles. For more on this matter, see [KAR], [SCH], [STO].

## Kurt Gödel (1906–1978)

Kurt Gödel had quite a happy childhood. He had rheumatic fever when he was six years old, but after he recovered life went on much as before.

Gödel entered the University of Vienna in 1923 still without having made a definite decision whether he wanted to specialize in mathematics or theoretical physics.

Gödel completed his doctoral dissertation under Hahn's supervision in 1929 submitting a thesis proving the completeness of the first order functional calculus. He became a member of the faculty of the University of Vienna in 1930.

Gödel is best known for his proof of "Gödel's Incompleteness Theorems." He proved fundamental results about axiomatic systems, showing in any axiomatic mathematical system there are propositions that cannot be proved or disproved within the axioms of the system.

Now 1933 was the year that Hitler came to power. At first this had no effect on Gödel's life in Vienna; he had little interest in politics. In 1934 Gödel gave a series of lectures at Princeton. However, Gödel suffered a nervous breakdown as he arrived back in Europe. He was treated by a psychiatrist and spent several months in a sanatorium.

Despite the health problems, Gödel's research was progressing well and he proved important results on the consistency of the axiom of choice with the other axioms of set theory in 1935.

He visited Göttingen in the summer of 1938, lecturing there on his set theory research. He returned to Vienna and married Adele Porkert in the autumn of 1938.

In 1940 Gödel arrived in the United States, becoming a U.S. citizen in 1948 (in fact he believed he had found an inconsistency in the United States Constitution, but the judge had more sense than to listen during his interview!). He was on the faculty of the Institute for Advanced Study from 1940 until his death. One of Gdel's closest friends at Princeton was Albert Einstein. They each had a high regard for the other and they spoke frequently.

He received the Einstein Award in 1951, and the National Medal of Science in 1974. He was a member of the National Academy of Sciences of the United States, a fellow of the Royal Society, a member of the Institute of France, and a fellow of the Royal Academy.

Towards the end of his life Gödel became convinced that he was being poisoned and, refusing to eat to avoid being poisoned, essentially starved himself to death.