Math 310
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Figure: This is your instructor.

The logical validity of the method of proof by mathematical induction is intimately bound up with the construction of the natural numbers, with ordinal arithmetic, and with the so-called well-ordering principle. However, the topic fits naturally into the present chapter. So we shall present and illustrate the method, and worry about its logical foundations later on. As with any good idea in mathematics, we shall be able to make it intuitively clear that the method is a valid and useful one. So no confusion should result.

Consider a statement $P(j)$ about the natural numbers. For example, the statement might be "The quantity $j^2 + 5j + 6$ is always even." If we wish to prove this statement, we might proceed as follows:

(1) Prove the statement $P(1)$.

(2) Prove that $P(j) \Rightarrow P(j+1)$ for every $j \in \{1, 2, \ldots\}$.

Let us apply the syllogism *modus ponendo ponens* to determine what we will have accomplished. We know $P(1)$ and, from (2) with $j = 1$, that $P(1) \Rightarrow P(2)$. We may therefore conclude $P(2)$. Now (2) with $j = 2$ says that $P(2) \Rightarrow P(3)$. We may then conclude $P(3)$. Continuing in this fashion, we may establish $P(j)$ for every natural number j.

Notice that this reasoning applies to any statement $P(j)$ for which we can establish (1) and (2) above. Thus (1) and (2) taken together constitute a method of proof. It is a method of establishing a statement $P(j)$ for every natural number j. The method is known as *proof by mathematical induction*.

It is worth enunciating the steps of the mathematical induction process in slightly different language (which should make it easier to remember):

Steps in an Inductive Proof

(a) Enunciate the inductive statement $P(j)$. This should be a simple, declarative sentence about the positive integer j.

(b) Verify the case $j = 1$.

(c) Verify that the case $P(j)$ implies the case $P(j + 1)$.

Example

Let us use the method of mathematical induction to prove that, for every natural number j, the number $j^2 + 5j + 6$ is even.

Solution:

(a) Our statement $P(j)$ is

The number $j^2 + 5j + 6$ is even.

[*Note:* Explicitly identifying $P(j)$ is more than a formality. *Always* record carefully what $P(j)$ is before proceeding.]

We now proceed in two steps:

(b) $P(1)$ is true. When $j = 1$ then

$$j^2 + 5j + 6 = 1^2 + 5 \cdot 1 + 6 = 12,$$

and this is certainly even. We have verified $P(1)$.

(c) $P(j) \Rightarrow P(j + 1)$. We are proving an implication in this step. We *assume* $P(j)$ and *use it* to establish $P(j + 1)$. Thus, we are assuming that

$$j^2 + 5j + 6 = 2m$$

for some natural number m. Then, to check $P(j + 1)$, we calculate

$$\begin{aligned} (j+1)^2 + 5(j+1) + 6 &= [j^2 + 2j + 1] + [5j + 5] + 6 \\ &= [j^2 + 5j + 6] + [2j + 6] \\ &= 2m + [2j + 6]. \end{aligned}$$

Notice that, in the last step, we have *used our hypothesis* that $j^2 + 5j + 6$ is even, that is that $j^2 + 5j + 6 = 2m$. Now the last line may be rewritten as

$$2(m + j + 3).$$

Thus, we see that $(j + 1)^2 + 5(j + 1) + 6$ is twice the natural number $m + j + 3$. In other words, $(j + 1)^2 + 5(j + 1) + 6$ is even. But that is the assertion $P(j + 1)$.

In summary, assuming the assertion $P(j)$ we have established the assertion $P(j + 1)$. That completes Step (c) of the method of mathematical induction. We conclude that $P(j)$ is true for every j.

Here is another example to illustrate the method of mathematical induction.

Proposition: *If j is any natural number, then*

$$1 + 2 + \cdots + j = \frac{j(j+1)}{2}.$$

Proof:
(a) The statement $P(j)$ is

$$1 + 2 + \cdots + j = \frac{j(j+1)}{2}.$$

Now let us follow the method of mathematical induction closely.

(b) $P(1)$ is true. The statement $P(1)$ is

$$1 = \frac{1(1+1)}{2}.$$

This is plainly true.

(c) $P(j) \Rightarrow P(j+1)$. We are proving an implication in this step. We *assume* $P(j)$ and *use it* to establish $P(j+1)$. Thus, we are assuming that

$$1 + 2 + \cdots + j = \frac{j(j+1)}{2}. \qquad (*)$$

Let us add the quantity $(j+1)$ to both sides of $(*)$. We obtain

$$1 + 2 + \cdots + j + (j+1) = \frac{j(j+1)}{2} + (j+1).$$

The left side of this last equation is exactly the left side of $P(j+1)$ that we are trying to establish. That is the motivation for our last step.

Now the right-hand side may be rewritten as

$$\frac{j(j+1) + 2(j+1)}{2}.$$

This simplifies to

$$\frac{(j+1)(j+2)}{2}.$$

In conclusion, we have established that

$$1 + 2 + \cdots + j + (j+1) = \frac{(j+1)(j+2)}{2}.$$

This is the statement $P(j+1)$.

Assuming the validity of $P(j)$, we have established the validity of $P(j+1)$. That completes the third step of the method of mathematical induction, and establishes $P(j)$ for all j. $\square$

Some problems are formulated in such a way that it is convenient to begin the mathematical induction with some value of j other than $j = 1$. The next example illustrates this notion:

Example

Let us prove that, for $j \geq 4$, we have the inequality

$$3^j > 2j^2 + 3j.$$

Solution:
(a) The statement $P(j)$ is

$$3^j > 2j^2 + 3j\,.$$

(b) $P(4)$ is true. Observe that the inequality is false for $n = 1, 2, 3$. However, for $n = 4$ it is certainly the case that

$$3^4 = 81 > 44 = 2 \cdot 4^2 + 3 \cdot 4.$$

(c) $P(j) \Rightarrow P(j+1)$. Now assume that $P(j)$ has been established and let us use it to prove $P(j+1)$. We are hypothesizing that

$$3^j > 2j^2 + 3j\,.$$

Multiplying both sides by 3 gives

$$3 \cdot 3^j > 3(2j^2 + 3j)$$

or

$$3^{j+1} > 6j^2 + 9j\,.$$

But now we have

$$\begin{aligned} 3^{j+1} &> 6j^2 + 9j \\ &= 2(j^2 + 2j + j) + (4j^2 + 3j) \\ &> 2(j^2 + 2j + 1) + (3j + 3) \\ &= 2(j+1)^2 + 3(j+1). \end{aligned}$$

This inequality is just $P(j+1)$, as we wished to establish. That completes step three of the mathematical induction, and therefore completes the proof.

Example

Recall that the sequence $\{f_n\}$ of *Fibonacci numbers* is

$$f_0 = 0, f_1 = 1, \ldots \ f_n = f_{n-1} + f_{n-2} \qquad \text{for all } n \geq 2\,.$$

The first several Fibonacci numbers are therefore $0, 1, 1, 2, 3,$ $5, 8, 13, 21, 34, 55, 89, \ldots$. We will prove now that every third Fibonacci number is even (note that the first Fibonacci number is f_0, so we are claiming that $f_3, f_6, f_9, \ldots$ are even). We do so by mathematical induction.

(a) The statement $P(j)$ is

$$f_{3j} \text{ is even.}$$

(b) First the case $j = 1$. We may verify by inspection that $f_3 = 2$.
(c) Now suppose that the assertion has been proved for j. We will verify it for $j + 1$. In fact we calculate that

$$\begin{aligned} f_{3(j+1)} &= f_{3j+3} \\ &= f_{3j+2} + f_{3j+1} \\ &= \big(f_{3j+1} + f_{3j}\big) + \big(f_{3j} + f_{3j-1}\big) \\ &= \Big(\big\{f_{3j} + f_{3j-1}\big\} + f_{3j}\Big) + \big(f_{3j} + f_{3j-1}\big) \\ &= 3f_{3j} + 2f_{3j-1}\,. \end{aligned}$$

Now we simply observe that, by the inductive hypothesis, f_{3j} is even. So the first term in the last line is even. Also, the second term has a factor of 2. So it is even. We conclude that $f_{3(j+1)}$ is even, and that completes the inductive step.

Example

Let us prove, by mathematical induction, that the sum of the first j positive, odd integers is equal to j^2.

We begin by writing the sum as

$$S_j = 1 + 3 + 5 + \cdots + \left[2(j-1)+1\right].$$

Notice that when $j = 1$ this just gives 1, with $j = 2$ it gives $1 + 3$, with $j = 3$ it gives $1 + 3 + 5$, and so forth.

(a) The statement $P(j)$ is

$$S_j = j^2 .$$

(b) The case $j = 1$ is the assertion $1 = 1^2$. That is certainly true.
(c) Now assume that the identity has been proved for some j. We will check it for $j + 1$. We have that

$$\begin{aligned} S_{j+1} &= 1 + 3 + 5 + \cdots + \left[2((j+1)-1)+1\right] \\ &= 1 + 3 + 5 + \cdots + \left[2j+1\right] \\ &= \left\{1 + 3 + 5 + \cdots + \left[2(j-1)+1\right]\right\} + \left[2j+1\right] \\ &= j^2 + \left[2j+1\right] \\ &= (j+1)^2 . \end{aligned}$$

In the penultimate equality we have used the inductive hypothesis. That completes the inductive step, and hence the proof.

We conclude this section by mentioning an alternative form of the mathematical induction paradigm which is sometimes called *complete mathematical induction* or *mathematical induction*.

Complete Mathematical Induction:

Let P be a function on the natural numbers. The steps are these.

(a) State $P(j)$;

(a) Prove $P(1)$;

(b) Prove that if $[P(\ell)$ is true for all $\ell \leq j]$ then $P(j+1)$ for every natural number j.

then $P(j)$ is true for every j.

It turns out that the complete mathematical induction principle is logically equivalent to the ordinary mathematical induction principle enunciated at the outset of this section. But in some instances strong mathematical induction is the more useful tool. An alternative terminology for complete mathematical induction is "the set formulation of mathematical induction."

Complete mathematical induction is sometimes more convenient, or more natural, to use than ordinary mathematical induction; it finds particular use in abstract algebra.

Example

Let us show, using complete mathematical induction, that every integer greater than 1 is either prime or the product of primes. [Here a prime number is an integer greater than 1 whose only factors are 1 and itself.]

For convenience we begin the mathematical induction process at the index 2 rather than at 1.

(a) Let $P(j)$ be the assertion "Either j is prime or j is the product of primes."
(b) Then $P(2)$ is plainly true since 2 is the first prime.

(c) Now assume that $P(\ell)$ is true for $2 \leq \ell \leq j$ and consider $P(j+1)$. If $j+1$ is prime, then we are done. If $j+1$ is not prime, then $j+1$ factors as $j+1 = k \cdot m$, where k, m are integers less than $j+1$, but at least 2. By the strong inductive hypothesis, each of k and m factors as a product of primes (or is itself a prime). Thus $j+1$ factors as a product of primes. The complete mathematical induction is done, and the proof is complete. $\square$