Math 310
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Figure: This is your instructor.

We give here a number of examples that illustrate proof techniques other than direct proof, proof by contradiction, and mathematical induction.

One obvious but powerful method for constructing a proof is to divide the question into *cases*. We illustrate with some examples.

Example

Let us show that every integer n which is a perfect cube is either (i) a multiple of 9, (ii) one less than a multiple of 9, or (iii) one more than a multiple of 9.

We consider three cases:

1. The integer n is the cube of a multiple of 3. So

$$n = (3j)^3 = 27j^3 = 9 \cdot (3j^3)\,.$$

In this case, it is clear that n is a multiple of 9.

2. The integer n is the cube of one less than a multiple of 3. Hence

$$n = (3j-1)^3 = 27j^3 - 27j^2 + 9j - 1 = 9(3j^3 - 3j^2 + j) - 1\,.$$

So we see that n is one less than a multiple of 9.

3. The integer n is the cube of one more than a multiple of 3. Therefore

$$n = (3j + 1)^3 = 27j^3 + 27j^2 + 9j + 1 = 9(3j^3 + 3j^2 + j) + 1 .$$

It is clear then that n is one more than a multiple of 9.

The three cases we have considered exhaust all the possibilities. So our proof is complete.

Example

We shall show that, if n is an integer, then

$$3n^2 + n + 14$$

is even. We divide into cases.

1. If n is even, then $n = 2k$ for some integer k. Thus

$$\begin{aligned} 3n^2 + n + 14 &= 3(2k)^2 + (2k) + 14 \\ &= 12k^2 + 2k + 14 \\ &= 2(6k^2 + k + 7) . \end{aligned}$$

   We see immediately that $3n^2 + n + 14$ is even.

2. If n is odd, then $n = 2k + 1$ for some integer k. Thus

$$\begin{aligned} 3n^2 + n + 14 &= 3(2k+1)^2 + (2k+1) + 14 \\ &= 3(4k^2 + 4k + 1) + (2k+1) + 14 \\ &= 12k^2 + 12k + 3 + 2k + 1 + 14 \\ &= 12k^2 + 14k + 18 \\ &= 2(6k^2 + 7k + 9)\,. \end{aligned}$$

We see immediately that $3n^2 + n + 14$ is even.

In both cases, which are exhaustive, our expression $3n^2 + n + 14$ is even. That completes the proof.

Example

We now verify the triangle inequality, which says that

$$|x + y| \leq |x| + |y|$$

for any real numbers x and y. [We note here that $|a| \leq \alpha$ if and only if $-\alpha \leq a \leq \alpha$. This fact will prove useful in the reasoning below.] To do so we divide the argument into cases:

Case 1: $x \geq 0$ and $y \geq 0$. In this case

$$|x + y| = x + y = |x| + |y| .$$

So our assertion is obvious.

Case 2: $x \geq 0$ and $y < 0$. In this case, we have

$$x + y \leq x + (-y) = |x| + |-y| = |x| + |y| . \qquad (*)$$

Furthermore,

$$-(x + y) = -x + (-y) \leq x + (-y) = |x| + |-y| = |x| + |y| .$$

Hence

$$x + y \geq -(|x| + |y|) . \qquad (**)$$

Combining $(*)$ and $(**)$ now gives

$$|x + y| \le |x| + |y| .$$

Case 3: $x < 0$ and $y \ge 0$. This case is identical to Case 2 (with the roles of x and y switched), and we omit the details.

Case 4: $x < 0$ and $y < 0$. Then

$$x + y \le (-x) + (-y) = |x| + |y| .$$

Also

$$-(x + y) = (-x) + (-y) = |x| + |y| .$$

In conclusion,

$$|x + y| \le |x| + |y| .$$

Combining all the cases—which certainly cover all the possibilities—we conclude that

$$|x + y| \leq |x| + |y| \qquad \text{for all } x \text{ and } y\,.$$

Example

Let us show that there exist irrational numbers a and b such that a^b is rational.

Let $\alpha = \sqrt{2}$ and $\beta = \sqrt{2}$. If α^β is rational, then we are done, using $a = \alpha$ and $b = \beta$. If α^β is irrational, then observe that

$$\left(\alpha^\beta\right)^{\sqrt{2}} = \alpha^{[\beta \cdot \sqrt{2}]} = \alpha^2 = [\sqrt{2}]^2 = 2\,.$$

Thus, with $a = \alpha^\beta$ and $b = \sqrt{2}$ we have found two irrational numbers a, b such that $a^b = 2$ is rational.

Proof by contraposition is closely related to proof by contradiction. But there is a subtle difference. If we are trying to prove that $P \Rightarrow Q$ by contradiction, then we deny Q but it is unclear what contradiction we are seeking. If we instead attempt a proof by contraposition, then we seek to prove $\sim Q \Rightarrow \sim P$. We assume $\sim Q$ and our goal is to prove $\sim P$. We illustrate with some examples.

Example

Let us show that, if n is an integer and n^2 is even, then n is even. We do so by contraposition. So suppose that n is odd. We shall then show that n^2 is odd.

The hypothesis then is that $n = 2k + 1$ for some integer k. Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 .$$

Thus we see explicitly that n^2 is odd, and our result is proved.

Example

We show that, if x, y are integers and $x + y$ is even then x and y have the same parity (i.e., either both are even or both are odd). We do so by proof by contraposition.

So suppose that x and y do *not* have the same parity. So one is even and one is odd. Say that x is even, so $x = 2j$ for some integer j, and y is odd, so $y = 2m + 1$ for some integer m. Then $x + y = (2j) + (2m + 1) = 2(j + m) + 1$. In conclusion, $x + y$ is odd. That gives the result.

Example

Let us prove that, if k is an integer and $3k+1$ is even, then k is odd. We do so by contraposition. Assume that k is even. So $k = 2m$ for some integer m. Then

$$3k + 1 = 3(2m) + 1 = 2(3m) + 1 \,,$$

which is clearly odd. That gives the result.

Example

Show that if there are 23 people in a room, then the odds are better than even that two of them have the same birthday.

Proof: Here by "same birthday" we mean birthday on the same day of the year. For convenience we shall assume that a year has 365 days.

The best strategy is to calculate the odds that *no two* of the people have the same birthday, and then to take complements.

Let us label the people $p_1, p_2, \ldots, p_{23}$. Then, assuming that none of the p_j have the same birthday, we see that p_1 can have his/her birthday on any of the 365 days in the year, p_2 can then have his/her birthday on any of the remaining 364 days, p_3 can have his/her birthday on any of the remaining 363 days, and so forth. So the number of different ways that these 23 people can all have different birthdays is

$$365 \cdot 364 \cdot 363 \cdots 345 \cdot 344 \cdot 343 \,.$$

On the other hand, the number of ways that birthdays could be distributed (with no restrictions) among 23 people is

$$\underbrace{365 \cdot 365 \cdot 365 \cdots 365}_{23 \text{ times}} = 365^{23} \, .$$

Thus the probability that these 23 people all have different birthdays is

$$p = \frac{365 \cdot 364 \cdot 363 \cdots 343}{365^{23}} .$$

A quick calculation with a pocket calculator shows that $p \sim 0.4927 < .5$. We see that the odds that 23 people will all have different birthdays is 0.4927. Thus the odds that at least two of them *will* have the same birthday is 0.5073, which is greater than one half. That is the desired result. $\square$

Example

Show that if there are six people in a room, then either three of them know each other or three of them do not know each other. [Here three people know each other if each of the three pairs has met. Three people do not know each other if each of the three pairs has *not* met.]

Proof: The tedious way to do this problem is to write out all possible "acquaintance assignments" for six people. That would take a good deal of time and effort, and would be woefully inelegant.

We now describe a more efficient, and more satisfying, strategy. Call one of the people Bob. There are five others. Either Bob knows three of them, or he does not know three of them.

Say that Bob knows three of the others. If any two of those three are acquainted, then those two and Bob form a mutually acquainted threesome. If no two of those three know each other, then those three are a mutually unacquainted threesome.

Now suppose that Bob does not know three of the others. If any two of those three are unacquainted, then those two and Bob form an unacquainted threesome. If all pairs among the three are instead acquainted, then those three form a mutually acquainted threesome.

We have covered all possibilities, and in every instance come up either with a mutually acquainted threesome or a mutually unacquainted threesome. That ends the proof. □

It may be worth knowing that five people is insufficient to guarantee either a mutually acquainted threesome or a mutually unacquainted threesome. We leave it to the reader to provide a suitable counterexample. It is quite difficult to determine the minimal number of people to solve the problem when "threesome" is replaced by "foursome." When "foursome" is replaced by five people, the problem is considered to be grossly intractable. This problem is a simple example from the mathematical subject known as Ramsey theory (see [GRS]).

Example

Jill is dealt a poker hand of five cards from a standard deck of 52. What is the probability that she holds four of a kind?

Remark: In order to solve this problem we need to note the following. If you want to choose k objects from among n objects, then there are n ways to choose the first object, $(n-1)$ ways to choose the second object, $(n-2)$ ways to choose the third object, down to $(n-k+1)$ ways to choose the kth object. And these objects can be chosen in any order. There are $k! = k \cdot (k-1) \cdot (k-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ ways to order k objects.

In conclusion, the number of ways to select k objects from among n is

$$\frac{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1} = \frac{n!}{k!(n-k)!} .$$

This fraction occurs so frequently in mathematics that it is denoted by the special symbol

$$\binom{n}{k} .$$

Proof: If the hand holds four aces, then the fifth card is any one of the other 48 cards. If the hand holds four kings, then the fifth card is any one of the other 48 cards. And so forth. So there are a total of

$$13 \times 48 = 624$$

possible hands with four of a kind. The total number of possible five-card hands is

$$\binom{52}{5} = 2598960\,.$$

Here we use the standard notation $\binom{n}{k}$ to denote the number of ways to choose k objects from among n. It is known that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}\,.$$

Therefore the probability of holding four of a kind is

$$p = \frac{624}{2598960} = 0.00024\,.$$

$\square$