Math 310
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Figure: This is your instructor.

Relations

Let S and T be sets. A *relation* on S and T is a subset of $S \times T$. If $\mathcal{R}$ is a relation, then we write either $(s, t) \in \mathcal{R}$ or sometimes $s \mathcal{R} t$ to indicate that s is related to t or that (s, t) is an element of the relation. We will also write $s \sim t$ when the relation being discussed is understood.

Example: Let $S = \mathbb{N}$, the natural numbers; and let $T = \mathbb{R}$, the real numbers. Define a relation $\mathcal{R}$ on S and T by $(s, t) \in \mathcal{R}$ if $s < \sqrt{t} < s + 1$. For instance, $(2, 5) \in \mathcal{R}$ because $\sqrt{5}$ lies between 2 and 3. Also $(4, 17) \in \mathcal{R}$ because $\sqrt{17}$ lies between 4 and 5. However, $(5, 10)$ does not lie in $\mathcal{R}$. Also $(3, \pi)$ does not lie in $\mathcal{R}$.

The *domain* of a relation $\mathcal{R}$ is the set of $s \in S$ such that there exists a $t \in T$ with $(s, t) \in \mathcal{R}$. The *image* of the relation is the set of $t \in T$ such that there exists an $s \in S$ with $(s, t) \in \mathcal{R}$. It is sometimes convenient to refer to the entire set T as the *range* of the relation $\mathcal{R}$. So we see that the image and the range are distinct. Some sources use the word "codomain" rather than "range."

Example: Let $S = \mathbb{N}$ and $T = \mathbb{N}$. Define a relation $\mathcal{R}$ on S and T by the condition $(s, t) \in \mathcal{R}$ if $s^2 < t$. Observe that, for any element $s \in \mathbb{N} = S$, the number $t = s^2 + 1$ satisfies $s^2 < t$. Therefore every $s \in S = \mathbb{N}$ is in the domain of the relation.

Now let us think about the image. The number $1 \in \mathbb{N} = T$ cannot be in the image since there is no element $s \in S = \mathbb{N}$ such that $s^2 < 1$. However, any element $t \in T$ that exceeds 1 satisfies $1^2 < t$. So $(1, t) \in \mathcal{R}$. Thus the image of $\mathcal{R}$ is the set $\{t \in \mathbb{N} : t \geq 2\}$.

Example: Let $S = \mathbb{N}$ and $T = \mathbb{N}$. Define a relation $\mathcal{R}$ on S and T by the condition $(s, t) \in \mathcal{R}$ if $s^2 + t^2$ is itself a perfect square. Then, for instance, $(3, 4) \in \mathcal{R}$, $(4, 3) \in \mathcal{R}$, $(12, 5) \in \mathcal{R}$, and $(5, 12) \in \mathcal{R}$. The number 1 is not in the domain of $\mathcal{R}$ since there is no natural number t such that $1^2 + t^2$ is a perfect square (if there were, this would mean that there are two perfect squares that differ by 1, and that is not the case). The number 2 is not in the domain of $\mathcal{R}$ for a similar reason. Likewise, 1 and 2 are not in the image of $\mathcal{R}$.

In fact, both the domain and image of $\mathcal{R}$ have infinitely many elements. This assertion will be explored in Exercise 4.56.

Many interesting relations arise for which S and T are the same set. Say that $S = T = A$. Then a relation on S and T is called simply a relation on A.

Example: Let $\mathbb{Z}$ be the integers. Let us define a relation $\mathcal{R}$ on $\mathbb{Z}$ by the condition $(s, t) \in \mathcal{R}$ if $s - t$ is divisible by 2. It is easy to see that both the domain and the image of this relation is $\mathbb{Z}$ itself. It is also worth noting that, if n is any integer, then the set of all elements related to n is either (i) the set of all even integers (if n is even) or (ii) the set of all odd integers (if n is odd).

Notice that the last relation created a division of the domain (=image) into two disjoint sets: the even integers and the odd integers. This was a special instance of an important type of relation that we now define.

Definition Let $\mathcal{R}$ be a relation on a set A. We say that $\mathcal{R}$ is an *equivalence relation* if the following properties hold:

$\mathcal{R}$ is reflexive: If $x \in A$, then $(x, x) \in \mathcal{R}$;

$\mathcal{R}$ is symmetric: If $(x, y) \in \mathcal{R}$, then $(y, x) \in \mathcal{R}$;

$\mathcal{R}$ is transitive: If $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$, then $(x, z) \in \mathcal{R}$.

Check for yourself that the relation described in Example 4.1.4 is in fact an equivalence relation. The most important property of equivalence relations is that which we indicated just before the definition and which we now enunciate formally:

Proposition *Let $\mathcal{R}$ be an equivalence relation on a set A. If $x \in A$, then define*

$$E_x \equiv \{y \in A : (x, y) \in \mathcal{R}\}.$$

We call the sets E_x the equivalence classes induced by the relation $\mathcal{R}$. If s and t are any two elements of A, then either $E_s \cap E_t = \emptyset$ or $E_s = E_t$.

In summary, the set A is the pairwise disjoint union of the equivalence classes induced by the equivalence relation $\mathcal{R}$.

Before we prove this proposition, let us discuss for a moment what it means. Clearly every element $a \in A$ is contained in some equivalence class, for a is contained in E_a itself. The proposition tells us that the set A is in fact the pairwise disjoint union of these equivalence classes. We say that the equivalence classes *partition* the set A.

For instance, in Example 4.1.4, the equivalence relation gives rise to two equivalence classes: the even integers $\mathcal{E}$ and the odd integers $\mathcal{O}$. Of course $\mathbb{Z} = \mathcal{E} \cup \mathcal{O}$ and $\mathcal{E} \cap \mathcal{O} = \emptyset$. We say that the equivalence relation *partitions* the universal set $\mathbb{Z}$ into two equivalence classes.

Notice that, in Example 4.1.4, if we pick any element $x \in \mathcal{E}$, then $E_x = \mathcal{E}$. Likewise, if we pick any element $y \in \mathcal{O}$, then $E_y = \mathcal{O}$.

Proof of the Proposition: Let $s, t \in A$ and suppose that $E_s \cap E_t \neq \emptyset$. It is our job to prove that $E_s = E_t$ (think for a moment about the truth table for "or" so that you understand that we are doing the right thing).

Since $E_s \cap E_t \neq \emptyset$, there is an element $x \in E_s \cap E_t$. Then $x \in E_s$. Therefore, by definition, $(s, x) \in \mathcal{R}$. Likewise, $x \in E_t$. Thus $(t, x) \in \mathcal{R}$. By symmetry, it follows that $(x, t) \in \mathcal{R}$. Now transitivity tells us that, since $(s, x) \in \mathcal{R}$ and $(x, t) \in \mathcal{R}$, then $(s, t) \in \mathcal{R}$.

If y is any element of E_t, then $(t, y) \in \mathcal{R}$. Transitivity now implies that since $(s, t) \in \mathcal{R}$ and $(t, y) \in \mathcal{R}$, then $(s, y) \in \mathcal{R}$. Thus $y \in E_s$. We have shown that every element of E_t is an element of E_s. Thus $E_t \subset E_s$.

Reversing the roles of s and t, we find that $E_s \subset E_t$. It follows that $E_s = E_t$. This is what we wished to prove. □

Example: Let A be the set of all people in the United States. If $x, y \in A$, then let us say that $(x, y) \in \mathcal{R}$ if x and y have the same surname (i.e., last name). Then $\mathcal{R}$ is an equivalence relation:

(i) $\mathcal{R}$ is reflexive since any person x has the same surname as his/her self.
(ii) $\mathcal{R}$ is symmetric since if x has the same surname as y, then y has the same surname as x.

(iii) $\mathcal{R}$ is transitive since if x has the same surname as y and y has the same surname as z, then x has the same surname as z.

Thus $\mathcal{R}$ is an equivalence relation. The equivalence classes are all those people with surname Smith, all those people with surname Herkimer, and so forth.

Example: Let S be the set of all residents of the United States. If $x, y \in S$, then let us say that x is related to y (that is, $x \sim y$) if x and y have at least one biological parent in common. It is easy to see that this relation is reflexive and symmetric. It is *not* transitive, as children of divorced parents know too well. What this tells us (mathematically) is that the proliferation of divorce in our society does *not* lead to well-defined families.

Example: Let S be the set of all residents of the United States. If $x, y \in S$, then let us say that x is related to y (that is, $x \sim y$) if x and y have *both* biological parents in common. It is easy to see that this relation is reflexive and symmetric. It is also transitive, since if A has the same Mom and Dad as B and B has the same Mom and Dad as C, then A, B, C are siblings and A has the same Mom and Dad as C. Contrast this situation with that in the last example.

What this tells us (mathematically) is that traditional families are defined by an equivalence relation.

Example: Let S be the set of integers and say that $x \sim y$ if $x \leq y$. This relation is clearly reflexive. It is *not* symmetric, as $3 \leq 5$ but $5 \not\leq 3$. You may check that it is transitive. But the failure of symmetry tells us that this is not an equivalence relation.

Example: Let f be a function with domain the real numbers and range the real numbers. We say that two numbers $a, b \in \mathbb{R}$ are related if $f(a) = f(b)$. This relation is clearly reflexive and symmetric. Also, if $f(a) = f(b)$ and $f(b) = f(c)$, then $f(a) = f(c)$. So the relation is also transitive, and it is therefore an equivalence relation. The equivalence classes are called *inverse images of points in the range*. For example, the set of all x such that $f(x) = 5$ is an equivalence class. It is the inverse image of 5.

Example: Let

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} .$$

Define

$$E_1 = \{1, 4, 7, 10\} \quad E_2 = \{2, 5, 8\} \quad E_3 = \{3, 6, 9\} .$$

Then the sets E_1, E_2, E_3 are pairwise disjoint, and their union is S. So these could be the equivalence classes for an equivalence relation, and in fact they are. What is that relation?

Say that $a \sim b$ if $b - a$ is divisible by 3. Check for yourself that this relation is reflexive, symmetric, and transitive. And verify that the equivalence class of 1 is E_1, the equivalence class of 2 is E_2, and the equivalence class of 3 is E_3.

This last example is an instance of a general phenomenon. If a set S is partitioned into subsets (pairwise disjoint sets whose union is S), then those subsets will be the equivalence classes for an equivalence relation.