# Math 310
# October 19, 2020 Lecture

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Figure: This is your instructor.

**Definition** Let $f$ and $g$ be functions with the same domain $S$ and the same range $T$. Assume that $T$ is a set in which the indicated arithmetic operations (below) makes sense. Then we define

(a) $(f+g)(x) = f(x) + g(x)$;

(b) $(f-g)(x) = f(x) - g(x)$;

(c) $(f \cdot g)(x) = f(x) \cdot g(x)$;

(d) $(f/g)(x) = f(x)/g(x)$ provided that $g(x) \neq 0$.

Notice that, in each of **(a)**–**(d)**, we are defining a *new function*—either $f + g$ or $f - g$ or $f \cdot g$ or $f/g$—in terms of the component functions $f$ and $g$. For practice, we shall express **(a)** in the language of ordered pairs. We ask you to do likewise with **(b)**, **(c)**, **(d)** in one of the exercises.

Let us consider part **(a)** in detail. Now $f$ is a collection of ordered pairs in $S \times T$ that satisfy the conditions for a function, and so is $g$. The function $f + g$ is given by

$$f + g = \{(s, t + t') : (s, t) \in f, (s, t') \in g\}.$$

Expressing the other combinations of $f$ and $g$ is quite similar, and you should be sure to do the corresponding exercise in the book.

**Example** Let $S = T = \mathbb{R}$. Define

$$f(x) = x^3 - x \qquad \text{and} \qquad g(x) = \sin(x^2).$$

Let us calculate $f + g,\ f - g,\ f \cdot g,\ f/g$.

Now

$$\begin{aligned}
(f+g)(x) &= (x^3 - x) + \sin(x^2) \\
(f-g)(x) &= (x^3 - x) - \sin(x^2) \\
(f \cdot g)(x) &= (x^3 - x) \cdot [\sin(x^2)] \\
(f/g)(x) &= (x^3 - x)/[\sin(x^2)] \quad \text{provided } x \neq \pm\sqrt{k\pi}\,, \\
& \qquad k \in \{0, 1, 2, \dots\}\,.
\end{aligned}$$

A more interesting, and more powerful, way to combine functions is through functional composition. Incidentally, in this discussion we will see the value of good mathematical notation.

**Definition** Let $f : S \to T$ be a function, and let $g : T \to U$ be a function. Then we define, for $s \in S$, the composite function

$$\big(g \circ f\big)(s) = g\big(f(s)\big). \qquad (*)$$

We call $g \circ f$ the *composition* of the functions $g$ and $f$.

Notice in this definition that the right-hand side of $(*)$ always makes sense because of the way that we have specified the domain and range of the component functions $f$ and $g$. In particular, we must have image $f \subset$ domain $g$ in order for the composition to make sense.

**Example** Let $f : \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}$ be given by $f(x) = x^4 + x^2 + 6$ and $g : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R}$ be given by $g(x) = \sqrt{x} - 4$. Notice that $f$ and $g$ fit the paradigm specified in the definition of composition of functions. Then

$$\begin{aligned} (g \circ f)(x) &= g\big(f(x)\big) \\ &= g\big(x^4 + x^2 + 6\big) \\ &= \sqrt{x^4 + x^2 + 6} - 4. \end{aligned}$$

Notice that $f \circ g$ also makes sense and is given by

$$\begin{aligned} (f \circ g)(x) &= f\bigl(g(x)\bigr) \\ &= f\bigl(\sqrt{x} - 4\bigr) \\ &= [\sqrt{x} - 4]^4 + [\sqrt{x} - 4]^2 + 6. \end{aligned}$$

It is important to understand that $f \circ g$ and $g \circ f$, when both make sense, will generally be different.

It is a good exercise in the ideas of this chapter to express the notion of functional composition in the language of ordered pairs. Thus let $f : S \to T$ be a function and $g : T \to U$ be a function. Then $f$ is a subset of $S \times T$ and $g$ is a subset of $T \times U$, both satisfying the two standard conditions for function. Now $g \circ f$ is a set of ordered pairs specified by

$$g \circ f = \Big\{(s,u) : s \in S, u \in U, \text{ and } \exists t \in T \text{ such that } (s,t) \in f \text{ and } (t,u) \in g\Big\}.$$

Take a moment to verify that this equation is consistent with the definition of functional composition that we gave earlier. Further note that $g \circ f$ is a set of ordered pairs from $S \times U$.

**Example** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin x^5$ and let $g : \{x \in \mathbb{R} : x \geq 1\} \to \mathbb{R}$ be given by $\sqrt[4]{x-1}$. We cannot consider $g \circ f$ because the range of $f$ (namely, the set $[-1, 1]$) does not lie in the domain of $g$. However, $f \circ g$ *does* make sense because the range of $g$ lies in the domain of $f$. And

$$\big(f \circ g\big)(x) = \sin\big[(x-1)^{5/4}\big].$$

**Definition** Let $S$ and $T$ be sets. Let $f : S \to T$ and $g : T \to S$. We say that $f$ and $g$ are *mutually inverse* provided that both $(f \circ g)(t) = t$ for all $t \in T$ and $(g \circ f)(s) = s$ for all $s \in S$. We write $g = f^{-1}$ or $f = g^{-1}$. We refer to the functions $f$ and $g$ as *invertible*; we call $g$ the *inverse* of $f$ and $f$ the *inverse* of $g$.

**Example** Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3 - 1$ and $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \sqrt[3]{x+1}$. Then

$$\begin{aligned}(f \circ g)(x) &= \left[\sqrt[3]{x+1}\right]^3 - 1 \\ &= (x+1) - 1 \\ &= x\end{aligned}$$

and

$$\begin{aligned}(g \circ f)(x) &= \sqrt[3]{(x^3 - 1) + 1} \\ &= \sqrt[3]{x^3} \\ &= x\end{aligned}$$

for all $x$. Thus $g = f^{-1}$ (or $f = g^{-1}$).

The idea of inverse function lends itself particularly well to the notation of ordered pairs. For $f : S \to T$ is inverse to $g : T \to S$ (and vice versa) provided that for every ordered pair $(s, t) \in f$ there is an ordered pair $(t, s) \in g$ and conversely.

Not every function has an inverse. For instance, let $f : S \to T$. Suppose that $f(s) = t$ and also that $f(s') = t$ with $s \neq s'$ (in other words, suppose that $f$ is not one-to-one). If $g : T \to S$, then $g(f(s)) = g(t) = g(f(s'))$ so it cannot be that both $g(f(s)) = s$ and $g(f(s')) = s'$. In other words, $f$ cannot have an inverse. We conclude that a function that *does* have an inverse must be one-to-one.

On the other hand, suppose that $t \in T$ has the property that there is no $s \in S$ with $f(s) = t$ (in other words, suppose that $f$ is not onto). Then, in particular, it could not be that $f(g(t)) = t$ for any function $g : T \to S$. So $f$ could not be invertible. We conclude that a function that *does* have an inverse must be onto.

**Example** Let $f : \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}$ be given by $f(x) = x^2$. Then $f$ is onto, but $f$ is not one-to-one. It follows that $f$ cannot have an inverse. And indeed it does not, for any attempt to produce an inverse function runs into the ambiguity that every positive number has two square roots.

Let $f : \{x \in \mathbb{R} : x \geq 0\} \to \mathbb{R}$ be given by $f(x) = x^2$. Then $f$ is one-to-one but $f$ is not onto. There certainly is a function $g : \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}$ such that $(g \circ f)(x) = x$ for all $x \in \{x \in \mathbb{R} : x \geq 0\}$ (namely $g(x) = \sqrt{|x|}$). But there is no function $g : \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}$ such that $(f \circ g)(x) = x$ for all $x$.

We have established that *if* $f : S \to T$ has an inverse, then $f$ must be one-to-one and onto. The converse is true too, and we leave the details for you to verify. A function $f : S \to T$ that is one-to-one and onto (and therefore invertible) is sometimes called a *set-theoretic isomorphism* or a *bijection*.

**Example** The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 5$ is one-to-one and onto (you should check this). Therefore it is invertible. To find the inverse, we consider the equation

$$f \circ f^{-1}(x) = x$$

or

$$f(f^{-1}(x)) = x\,.$$

We may write this out as

$$3f^{-1}(x) + 5 = x\,.$$

Solving for $f^{-1}(x)$ gives

$$f^{-1}(x) = \frac{x-5}{3}\,.$$

Check for yourself that $f \circ f^{-1}(x) = x$ and $f^{-1} \circ f(x) = x$.

**Example** The function $f(x) = x^3 + 1$ is one-to-one and onto. Therefore it is invertible. To find the inverse, we consider the equation

$$f \circ f^{-1}(x) = x$$

or

$$f(f^{-1}(x)) = x \,.$$

We may write this out as

$$[f^{-1}(x)]^3 + 1 = x \,.$$

Solving for $f^{-1}(x)$ gives

$$f^{-1}(x) = \sqrt[3]{x-1} \,.$$

**Example** In this example we use some ideas from calculus.

The function $f(x) = x^3 + x$ satisfies $f'(x) = 3x^2 + 1 > 0$ for every $x$. Therefore $f$ is strictly increasing. So it is one-to-one. Also $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$. So $f$ is onto. Therefore $f$ is invertible.

It would be quite difficult to solve the equation

$$f \circ f^{-1}(x) = x ,$$

and we shall not attempt to do so.

**Example** The function $f : \mathbb{R} \to \mathbb{R}$ that is given by $f(x) = x^3$ is a bijection. You should check the details of this assertion for yourself. The inverse of this function $f$ is the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^{1/3}$.

We leave it as an exercise for you to verify that the composition of two bijections (when the composition makes sense) is a bijection.