Math 310
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Steven G. Krantz

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Figure: This is your instructor.

In what follows, we will consistently use some important and universally recognized terminology. An infinite set S is said to be *countable* if it has the same cardinality as the natural number $\mathbb{N}$. If an infinite set is *not* countable, that is if it does *not* have a bijection with the natural numbers $\mathbb{N}$, then it is said to be *uncountable*. Every infinite set is either countable or uncountable. One of our big jobs in this section of the book is to learn to recognize countable and uncountable sets.

Now it is time to look at some specific examples.

Example Let $\mathcal{E}$ be the set of all even integers and $\mathcal{O}$ the set of all odd integers. Then

$$\text{card}(\mathcal{E}) = \text{card}(\mathcal{O}).$$

Indeed, the function

$$f(j) = j + 1$$

is a bijection from $\mathcal{E}$ to $\mathcal{O}$.

Example Let $\mathcal{E}$ be the set of even integers. Then

$$\mathrm{card}(\mathcal{E}) = \mathrm{card}(\mathbb{Z})\,.$$

The function

$$g(j) = j/2$$

gives the bijection. Thus $\mathrm{card}(\mathcal{E}) = \mathrm{card}(\mathbb{Z})$.

This last example is a bit surprising, for it shows that a set (namely, $\mathbb{Z}$, the integers) can be put in one-to-one correspondence with a proper subset (namely $\mathcal{E}$, the even integers) of itself. In other words, $\mathbb{Z}$ has the same cardinality (that is, the same number of elements) as a proper subset of itself. This phenomenon is impossible for finite sets.

Example We have

$$\mathrm{card}(\mathbb{Z}) = \mathrm{card}(\mathbb{N}).$$

We define the function f from $\mathbb{Z}$ to $\mathbb{N}$ as follows:

- $f(j) = -(2j + 1)$ if j is negative
- $f(j) = 2j + 2$ if j is positive or zero

The values that f takes on the negative integers are $1, 3, 5, \ldots,$ on the positive integers are $4, 6, 8, \ldots,$ and $f(0) = 2$. Thus f is one-to-one and onto.

By putting together the preceding examples, we see that the set of even integers, the set of odd integers, and the set of all integers are countable sets.

Example The set of all ordered pairs of positive integers

$$S = \mathbb{N} \times \mathbb{N} = \{(j, k) : j, k \in \mathbb{N}\}$$

is countable.

To see this, we will use the Schroeder–Bernstein theorem. The function

$$f(j) = (j, 1)$$

is a one-to-one function from $\mathbb{N}$ to S. Also, the function

$$g(j, k) = j \cdot 10^{j+k} + k$$

is a function from S to $\mathbb{N}$. Let n be the number of digits in the number k. Notice that $g(j, k)$ is obtained by writing the digits of j, followed by $j + k - n$ zeros, then followed by the digits of k. For instance,

$$g(23, 714) = 23\underbrace{000\ldots000}_{734}714,$$

where there are $23 + 714 - 3 = 734$ zeros between the 3 and the 7. It is clear that g is one-to-one. By the Schroeder–Bernstein theorem, S and $\mathbb{N}$ have the same cardinality; hence S is countable.

There are other ways to handle the last example, and we shall explore them in the exercises.

Since there is a bijection f of the set of *all* integers $\mathbb{Z}$ with the set $\mathbb{N}$, it follows from the last example that the set $\mathbb{Z} \times \mathbb{Z}$ of all pairs of integers (positive *and* negative) is countable. Let f be the function from the example that showed that $\mathbb{Z}$ is countable. Then the map $\big(f \times f\big)(x, y) = (f(x), f(y))$ is a bijection of $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N} \times \mathbb{N}$. Let h be the bijection, provided by the last example, from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. Then $h \circ \big(f \times f\big)$ is a bijection of $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{N}$.

Notice that the word “countable” is a good descriptive word: if S is a countable set, then we can think of S as having a first element s_1 (the one corresponding to $1 \in \mathbb{N}$), a second element s_2 (the one corresponding to $2 \in \mathbb{N}$), and so forth. Thus we write $S = \{s_1, s_2, \ldots\}$.

Definition A set S is called *finite* if it is either empty or else there is a bijection of S with a set of the form $I_n \equiv \{1, 2, \ldots, n\}$ for some positive integer n. If S is not empty and if no such bijection exists, then the set is called *infinite*.

Remark The empty set is a finite set, but not one of any particular interest. Nevertheless we must account for it, so we include it explicitly in the definition of "finite set."

In some treatments, a different approach is taken to the concepts of "finite" and "infinite" sets. In fact, one defines an infinite set to be one which can be put in one-to-one correspondence with a proper subset of itself. For instance, an earlier example shows that the set $\mathbb{Z}$ of all integers can be put in one-to-one correspondence with the set $\mathcal{E}$ of all even integers (and of course $\mathcal{E}$ is a proper subset of $\mathbb{Z}$). By contrast, a finite set *cannot* be put in one-to-one correspondence with a proper subset of itself. This last assertion amounts to verifying that I_k cannot be put in one-to-one correspondence with I_n when $k > n$. But any function $f : I_k \to I_n$ would be sending k letters to n mailboxes. By the pigeonhole principle, two letters would have to land in the same box. So the function cannot be one-to-one. Thus I_k and I_n do *not* have the same cardinality.

An important property of the natural numbers $\mathbb{N}$ is that any subset $S \subset \mathbb{N}$ has a least element. See the discussion in an earlier lecture and also later on. This is known as the Well-Ordering Principle, and is studied in a course on logic. In the present chapter, we take the properties of the natural numbers as given (see our later treatment of the natural numbers). We use some of these properties in the next proposition.

Proposition *If S is a countable set and R is a subset of S, then either R is empty or R is finite or R is countable.*

Proof: Assume that R is not empty.

Write $S = \{s_1, s_2, \dots\}$. Let j_1 be the least positive integer such that $s_{j_1} \in R$. Let j_2 be the least integer following j_1 such that $s_{j_2} \in R$. Continue in this fashion. If the process terminates at the n^{th} step, then R is finite and has n elements.

If the process does not terminate, then we obtain an enumeration of the elements of R :

$$\begin{gathered} 1 \longleftrightarrow s_{j_1} \\ 2 \longleftrightarrow s_{j_2} \\ \cdots \end{gathered}$$

etc.

All elements of R are enumerated in this fashion since $j_\ell \geq \ell$. Therefore R is countable. $\square$

A set is called *countable* if it is countably infinite, that is, if it can be put in one-to-one correspondence with the natural numbers $\mathbb{N}$. A set is called *denumerable* if it is either empty or finite or countable. In actual practice, mathematicians use the word "countable" to describe sets that are either finite or countable. In other words, they use the word "countable" interchangeably with the word "denumerable."

The set $\mathbb{Q}$ of all rational numbers consists of all expressions

$$\frac{a}{b},$$

where a and b are integers and $b \neq 0$. Thus $\mathbb{Q}$ can be identified with the set of all pairs (a, b) of integers. After discarding duplicates, such as $\frac{2}{4} = \frac{1}{2}$, and using the earlier discussion to the effect that $\mathbb{Z} \times \mathbb{Z}$ is countable, we find that the set $\mathbb{Q}$ is countable. We shall deal with the rational number system in a much more precise manner later on.

Theorem *Let S_1, S_2 be countable sets. Set $\mathcal{S} = S_1 \cup S_2$. Then $\mathcal{S}$ is countable.*

Proof: *Let us write*

$$S_1 = \{s_1^1, s_2^1, \ldots\}$$

$$S_2 = \{s_1^2, s_2^2, \ldots\}.$$

If $S_1 \cap S_2 = \emptyset$, then the function

$$s_j^k \mapsto (j,k)$$

is a bijection of $\mathcal{S}$ with a subset of $\{(j,k) : j,k \in \mathbb{N}\}$. We proved earlier that the set of ordered pairs of elements of $\mathbb{N}$ is countable. By an earlier result, $\mathcal{S}$ is countable as well.

If there exist elements which are common to S_1, S_2, then discard any duplicates. The same argument shows that $\mathcal{S}$ is countable. $\square$

Proposition *If S and T are each countable sets, then so is*

$$S \times T = \{(s,t) : s \in S, t \in T\} .$$

Proof: *Since S is countable, there is a bijection f from S to $\mathbb{N}$. Likewise there is a bijection g from T to $\mathbb{N}$. Therefore the function*

$$(f \times g)(s,t) = (f(s), g(t))$$

is a bijection of $S \times T$ with $\mathbb{N} \times \mathbb{N}$, the set of ordered pairs of positive integers. But we saw in an earlier example that $\mathbb{N} \times \mathbb{N}$ is a countable set. Hence so is $S \times T$. $\square$

Corollary *If $S_1, S_2, \ldots, S_k$ are each countable sets, then so is the set*

$$\mathbf{S}_1 \times \mathbf{S}_2 \times \cdots \times \mathbf{S_k} = \{(\mathbf{s_1}, \ldots, \mathbf{s_k}) : \mathbf{s_1} \in \mathbf{S_1}, \ldots, \mathbf{s_k} \in \mathbf{S_k}\}$$

consisting of all ordered $\mathbf{k}$*-tuples* $(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s_k})$ *with* $\mathbf{s_j} \in \mathbf{S_j}$*.*
Proof: *We may think of* $\mathbf{S}_1 \times \mathbf{S}_2 \times \mathbf{S}_3$ *as* $(\mathbf{S}_1 \times \mathbf{S}_2) \times \mathbf{S}_3$*. Since* $\mathbf{S}_1 \times \mathbf{S}_2$ *is countable (by the Proposition) and* $\mathbf{S}_3$ *is countable, then so is* $(\mathbf{S}_1 \times \mathbf{S}_2) \times \mathbf{S}_3 = \mathbf{S}_1 \times \mathbf{S}_2 \times \mathbf{S}_3$ *countable. Continuing in this fashion (i.e., inductively), we can see that any finite product of countable sets is also a countable set.* $\square$

Corollary *The countable union of countable sets is countable.*
Proof: *Let* $\mathsf{A}_1, \mathsf{A}_2, \ldots$ *each be countable sets. If the elements of* A_j *are enumerated as* $\{\mathsf{a}_\mathsf{k}^\mathsf{j}\}_{\mathsf{k}=1}^{\infty}$ *and if the sets* A_j *are pairwise disjoint, then the correspondence*

$$\mathsf{a}_\mathsf{k}^\mathsf{j} \longleftrightarrow (\mathsf{j}, \mathsf{k})$$

is one-to-one between the union of the sets A_j *and the countable set* $\mathbb{N} \times \mathbb{N}$*. This proves the result when the sets* A_j *have no common element. If some of the* A_j *have elements in common, then we discard duplicates in the union and use the earlier argument.* $\square$