Math 310
October 28, 2020 Lecture

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Figure: This is your instructor.

This lecture considers a fun treatment of Cantor's ideas that is due to the great German mathematician David Hilbert.

This somewhat frivolous, but absolutely mathematically rigorous, discussion gives a nice illustration of the concept of countable set.

In 1934 the eminent mathematician David Hilbert, in a public lecture, proposed the idea of "Hotel Infinity." It is a fascinating notion, and we describe it here. The charming and entertaining book [GAM] explores the idea further.

Imagine a hotel with infinitely many rooms. For convenience, these rooms are numbered $1, 2, 3, \ldots$. See the figure.

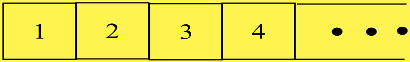

Figure: The hotel infinity.

Now suppose that, on a given evening, all the rooms are occupied. But a new guest, who does *not* have a reservation, shows up. What is the hotel manager to do?

If this were an ordinary hotel with finitely many rooms, then the manager would be stuck. She would have no room for the new guest, and would have to send her away. But instead she has infinitely many rooms, so she can do something creative:

> She moves the guest in Room 1 to Room 2, and the guest in Room 2 to Room 3, and the guest in Room 3 to Room 4, and so on. In this way every one of the pre-existing guests will still have a room. But Room 1 will be freed up and she can put the new guest in Room 1.

Miraculous, no??

The same thing would work if 5 new guests were to show up. Here is how she would proceed:

> She moves the guest in Room 1 to Room 6, and the guest in Room 2 to Room 7, and the guest in Room 3 to Room 8, and so on. In this way every one of the pre-existing guests will still have a room. But Rooms 1 through 5 will be freed up and she can put the new guests in Rooms 1 through 5.

Now what if infinitely many new guests show up? That is more of a challenge. But the hotel manager is a talented mathematical thinker, and she comes up with this solution:

> She moves the guest in Room 1 to Room 2. Then she moves the guest in Room 2 to Room 4. Next she moves the guest in Room 3 to Room 6. In general, she moves the guest in Room k to Room $2k$. This frees up Rooms $1, 3, 5, 7, \ldots$. So infinitely many rooms have been freed up, and she can accommodate infinitely many new guests.

The devil is working to try to trip up our intrepid hotel manager. So he comes up with the following situation. He brings in infinitely many buses, and each bus has infinitely many passengers. He wants her to provide accommodations for *all* of these new guests. How does she do it?

First, she moves the guest in Room 1 to Room 2. Then she moves the guest in Room 2 to Room 4. Next she moves the guest in Room 3 to Room 6. And so on.

This frees up all the odd-numbered rooms. But she needs to accommodate infinitely many collections of infinitely many people. How can she do it?

She takes the people from the first bus and puts them in the rooms numbered

$$3^1 \ , 3^2 \ , \ 3^3 \ , \ldots$$

So the first bus passengers all go into rooms whose numbers are powers of 3. These are of course odd-numbered rooms, so they are vacant.

Next, she takes the people from the second bus and puts them in the rooms numbered

$$5^1, 5^2, 5^3, \ldots$$

So the second bus passengers all go into rooms whose numbers are powers of 5. These are of course odd-numbered rooms, and they are different from the rooms with numbers that are powers of 3. So these rooms are vacant, and can accommodate the passengers from bus number 2.

For the next step, she takes the people from the third bus and puts them in the rooms numbered

$$7^1 \ , 7^2 \ , \ 7^3 \ , \ldots$$

So the third bus passengers all go into rooms whose numbers are powers of 7. These are of course odd-numbered rooms, and they are different from the rooms with numbers that are powers of 3 or powers of 5. So these rooms are vacant, and can accommodate the passengers from bus number 3.

The hotel manager continues in this fashion, placing passengers from bus k into rooms with numbers that are powers of the kth odd prime number.

Take particular note that, for the fourth bus, we do not put the guests into the rooms numbered 9^1, 9^2, 9^3, ... Because these are actually powers of 3, so these rooms would have been filled when we accommodate the guests from the first bus. It is important that we only use rooms with numbers that are *powers of the odd prime numbers*. So, for the fourth bus, we use the rooms with numbers 11^1, 11^2, 11^3, ...

The method that we have described for infinitely many buses each with infinitely many passengers certainly works, but it also leaves infinitely many rooms empty. For example, the room with number $15 = 5 \cdot 3$ will be empty. But this may be convenient for the hotel.

In fact Georg Cantor (1845–1918) was the man who developed techniques for handling infinite sets. He showed that infinite sets come in different sizes, and he created a calculus for manipulating infinite sets. He is remembered today as one of the great mathematicians of the last hundred years. Hilbert's Hotel Infinity was in fact a commentary on and an appreciation of Cantor's work.

Exercise: Imagine infinitely many buses showing up, each with infinitely many passengers, but these people are only willing to stay in rooms with even numbers. How can the hotel manager accommodate these new guests?