Math 310
October 30, 2020 Lecture

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Figure: This is your instructor.

In a rigorous treatment of set theory such as one encounters in an advanced course in logic, there is a detailed consideration of the *axioms* of set theory. As indicated in Chapter 1, this is part and parcel of the way that mathematics is done "for the record." It is the way that we insure that no logical inconsistencies will arise in our future work, and it is the way that we lay the ground rules for our subject.

The present book constitutes your first exposure to rigorous mathematical thinking, and it would be somewhat overwhelming for you to have to wade through a thorough treatment of the axioms of set theory at this time. We have instead been studying a "naive" treatment of set theory. Nonetheless, we should like now briefly to treat the formal axioms and to discuss some of the more significant axioms that arise frequently in advanced mathematics.

The Axioms of Zermelo–Fraenkel Set Theory

Axiom of Extensionality

$$\big[\forall x, (x \in A \iff x \in B)\big] \Rightarrow (A = B)$$

Sum Axiom Given a collection $\mathcal{S}$ of sets:

$$\exists C\ \forall x, \Big(x \in C \iff \exists B, (x \in B \wedge B \in \mathcal{S})\Big)$$

Power Set Axiom Given a set A:

$$\exists B\ \forall C, (C \in B \iff C \subset A)$$

Axiom of Regularity

$$A \neq \emptyset \Rightarrow \exists x, \left[x \in A \wedge \forall y, (y \in x \Rightarrow y \notin A)\right]$$

Axiom for Cardinals If A is a set, then let $\mathcal{K}(A)$ denote the set of all sets that are set-theoretically isomorphic to A. The set $\mathcal{K}(A)$ is sometimes called the *cardinality of A*, or the *cardinal number* corresponding to A. Then

$$\mathcal{K}(A) = \mathcal{K}(B) \iff \text{card}(A) \equiv \text{card}(B),$$

where $\equiv$ means set-theoretic isomorphism.

Axiom of Infinity

$$\exists A, (\emptyset \in A \wedge \forall B, [B \in A \Rightarrow B \cup \{B\} \in A])$$

Axiom Schema of Replacement Let $P(x, y)$ be a property of x, y. If

$$\forall x \; \forall y \; \forall z, \left[\{x \in A \wedge P(x, y) \wedge P(x, z)\} \Rightarrow \{\mathbf{y} = \mathbf{z}\} \right]$$

then

$$\exists B \; \forall y, [y \in B \Leftrightarrow \exists x, (x \in A \wedge P(x, y))].$$

Axiom of Choice For any set A, there is a function $f : \mathcal{P}(A) \to A$ such that, for any nonempty subset B of A, we have $f(B) \in B$.

The advantage of the way that we have stated the axioms here is that the actual statements do not involve English words (which are subject to misunderstanding). The statements are also brief. But it requires some effort to understand what they say. To aid in this process, we now give a brief, informal description of what each axiom says.

The Axiom of Extensionality mandates that two sets are equal precisely when they have the same elements. In our treatment in Section 3.1, we took this property as a definition.

The Sum Axiom specifies that if we are given a collection of sets $\mathcal{S} = \{S_\alpha\}_{\alpha \in A}$, then the union set $C = \cup_{\alpha \in A} S_\alpha$ exists. Again, in the interest of simplicity we treated this idea in Chapter 3 with a definition. Lurking in the background here is the fact (to be explored briefly at the end of this section) that we cannot allow the existence of sets that are too large. If we do, then certain paradoxes result. Thus certain of our set theory axioms specify the ways in which we are allowed to form new sets.

The Power Set Axiom says that if we are given a set A, then its power set $\mathcal{P}(A)$ also exists. The comments in the last paragraph about creating new sets also apply here.

The Axiom of Regularity is slightly more subtle. Consider the set S of all sets that can be described with fewer than 50 words. Then S is an element of itself. Such considerations can lead to nasty paradoxes (see Russell's Paradox at the end of the section). For this reason, we want to rule out sets that are elements of themselves. As an exercise, you may try to prove, using the Axiom of Regularity, that if A is a set, then $A \notin A$.

The Axiom for Cardinals addresses another existence problem. Given a set S, we wish to consider the set of sets each of which has the same cardinality as S. This is an equivalence class (with the equivalence relation being set-theoretic isomorphism), but we do not know in advance that the equivalence class exists. The Axiom for Cardinals mandates just that. Again, this is one of our allowed methods for creating new (possibly large) sets. Some versions of set theory disallow extremely large sets by not including the Axiom for Cardinals.

The Axiom of Infinity specifies the existence of a set that is set-theoretically isomorphic to a proper subset of itself. We know from Section 4.5 that this is equivalent to postulating the existence of an infinite set. In fact, our formulation of the Axiom of Infinity is very closely related to the classical construction of ordinal numbers and, more specifically, of the natural numbers themselves. See the discussion in Chapter 6.

The Axiom of Replacement is perhaps the most technical axiom, but it is also the one that is used most often. It is our device for passing from the general to the specific. It is our means of specifying a set as the collection of objects that satisfy a certain property. The first line of the axiom specifies that P is a property that can be treated. The second line specifies that there exists a set B that is the set of elements that satisfy property P.

The Axiom of Choice is perhaps the axiom that has been the richest source of ideas, and also of confusion, in all of set theory. Its intuitive statement is simplicity itself: Let S be a set. Then there is a function $f : \mathcal{P}(S) \to S$ that assigns to each nonempty subset of S an element of itself. While quite a plausible axiom, it is a metatheorem of mathematical logic that it is impossible to specify the function that assigns to each nonempty subset of the reals an element of itself. By contrast, with $S = \mathbb{N}$, the natural numbers, we can do it: if $A \subset \mathbb{N}$, then let $f(A)$ be the least element of A. We shall prove in our treatment of the natural numbers in Section 6.1 that every subset of the natural numbers does indeed *have* a least element. So this choice function f makes good sense and does the job. Think for a moment why a function defined in this way will not work for the real numbers.

We shall spend the next section discussing the Axiom of Choice and its consequences.

We close this section with a brief mention of Russell's Paradox. There are many paradoxes that fall under this rubric, and Russell's was not the first. A nice discussion of these matters appears in [SUP, Ch.1]. See also [JEC].

Russell's Paradox: Let S be the set of all sets which are not elements of themselves (most sets fit this description, although we gave an example of a set which *is* an element of itself in the discussion of the Axiom of Regularity). Is S an element of itself?

If S is an element of itself then, by definition, S is not an element of itself. On the other hand, if S is not an element of itself then it must be an element of itself (since S has as elements *all* sets which are not elements of themselves).

Thus S can neither be an element of itself nor not be an element of itself.

Russell's paradox was first communicated to Frege just weeks before the latter's definitive book on the foundations of set theory was to appear in print. The paradox called into question Frege's, and everyone else's, approach to set theory. It is because of Russell's Paradox that modern versions of set theory, such as that given by the eight axioms above, set very careful restrictions on which sets we may construct and consider. In particular, the set S of Russell's Paradox is too large to exist in the version of Zermelo–Fraenkel set theory that was discussed earlier in this section. The concept of "class" has been developed to deal with objects that are too large to be considered a part of set theory. See [FRA] or [COH] for a discussion of classes.

We close this section by summarizing the discussion: In a formal treatment of set theory, one begins with the eight axioms specified above and with the empty set and systematically develops the properties of sets. This is done, for instance, in [SUP]. In the present text, we have given an intuitive treatment of set theory that parallels the classical, rigorous treatment but is more accessible.