Math 310
November 4, 2020 Lecture

Steven G. Krantz

October 30, 2020

Figure: This is your instructor.

Independence and Consistency

We now turn to a brief and informal discussion of independence and consistency of the axioms of set theory. It is part of the spirit of the axiomatic method to have *as few axioms as possible*. For instance, it would be silly to have axioms that looked like this

1. $A \Rightarrow B$
2. $B \Rightarrow C$
3. $A \Rightarrow C$

because Axiom 3 follows logically from the other two axioms. It does not need to be enunciated as an axiom (although, conceivably, it might be useful later on to record it as a proposition). In this circumstance, we say that Axiom 3 is *not independent* of the other axioms.

Now the example that we have just presented is misleadingly simple. On the face of it, the axioms of set theory that we presented in the Section 5.1 all say something different. But it is conceivable, is it not, that after many pages of argument one might show that the first, third, and fifth axioms imply the sixth? How could we establish that such an eventuality cannot occur?

The question posed in the last paragraph was not considered formally until the twentieth century. The principal method that has evolved for showing that statement X is independent of axioms $A_1, A_2, \ldots, A_k$ is as follows.

1. One constructs a mathematical entity (such as a number system, or a geometry, or a version of set theory) in which $A_1, A_2, \ldots, A_k$ are true and X is also true.
2. One constructs a mathematical entity (such as a number system, or a geometry, or a version of set theory) in which $A_1, A_2, \ldots, A_k$ are true but X is false.

A moment's thought shows that $\sim X$ could not be proved as a theorem in the axiom system specified by $A_1, \ldots, A_k$, since (Statement **1**) we have constructed an instance of the axioms in which X is true (i.e., $\sim X$ fails). Likewise, statement X could not be proved as a theorem in the axiom system specified by $A_1 \ldots, A_k$ since (Statement **2**) we have constructed an instance of the axioms in which X fails (i.e., $\sim X$ is true).

All this may sound rather abstract, so let us consider a specific example. Euclidean geometry has five axioms. The first four are these:

P1. Through any pair of distinct points there passes a line.

P2. For each segment $\overline{AB}$ and each segment $\overline{CD}$ there is a unique point E (on the line determined by A and B) such that B is between A and E and the segment $\overline{CD}$ is congruent to $\overline{BE}$ (see the figure).

P3. For each point C and each point A distinct from C there exists a circle with center C and radius CA (see the next figure).

P4. All right angles are congruent (see the next figure).

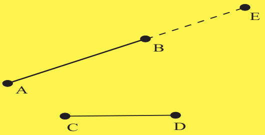
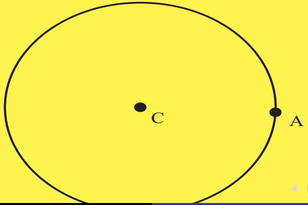


Figure: Euclid's second postulate.


C
A

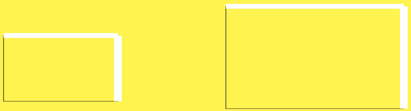
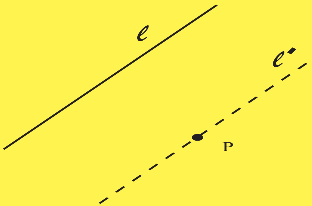


Figure: Euclid's fourth postulate.

These are the standard four axioms that give our Euclidean conception of geometry. The fifth axiom, a topic of intense study for two thousand years, is the so-called parallel postulate (in Playfair's formulation):

P5. For each line ℓ and each point P that does not lie on ℓ there is a unique line ℓ' through P such that ℓ' is parallel to ℓ (see the next figure).


ℓ
ℓ'
P

The "parallel postulate" of Euclidean geometry specifies that if ℓ is a line in the plane and P is a point not on that line, then there is one and only one line ℓ' passing through P that is parallel to ℓ. Classically, vigorous attempts were made to prove the parallel postulate as a *theorem* from the first four basic axioms of geometry. However, in the early nineteenth century, Nikolai Ivanovich Lobachevski (1793–1856) and János Bolyai (1775–1856) constructed a geometry in which the first four of the axioms of Euclid are satisfied yet the parallel postulate *fails* (see Section 11.2).

Our standard (Euclidean/Cartesian) geometry is one in which all the axioms of Euclid are satisfied and the parallel postulate *holds*. The conclusion from the last two sentences is that the parallel postulate is *independent* of the other axioms of geometry. It cannot be proved as a theorem, nor can its negation be proved. What is true is that *either* the parallel postulate P *or* its negation $\sim P$ can be adjoined to the other axioms to form a valid geometry. More will be said about these different geometries in Chapter 11.

Besides independence, the other big issue in setting up an axiomatic system is *consistency*. One could write down several plausible looking axioms which seem logical and start doing mathematics based on those axioms. Then one could wake up one day two years from now only to prove that $1 = 2$ or that $5 < 5$. This would be catastrophic.

How can we guarantee that an eventuality such as that described in the last paragraph can never happen? The answer comes by way of model theory. Think about how real life works. Two hundred or more years ago, the best scientists of the day gave cogent arguments that there could never be a heavier-than-air machine that flies. Today there are airplanes made of metal, that weigh many tons, and that fly. No matter how good the two-hundred-year-old (strictly philosophical) arguments are, and no matter how persuasive, the existence of metal airplanes lays the matter to rest.

Now let us get a bit more mathematical. Suppose I say to you that there is a number system containing the reals in which every polynomial equation has a root. When I say this, you should keep in mind that the polynomial $x^2 + 1 = 0$ has no *real roots*. We could debate this matter, from a philosophical point of view, at length. But in Chapter 6 we shall *construct* a number system, called the complex numbers, in which this polynomial, indeed any polynomial, has a root.

The important word in the last paragraph is "construct." There is no value in saying "Yes, there is this number system that I dreamed up, and it contains a square root for -1 and other mysterious artifacts." We must construct the number system from mathematical tools at hand. This whole circle of ideas will be treated in greater detail in Chapter 6.

Nikolai Lobachevski (1793–1856)

Nikolai Ivanovich Lobachevski was one of three sons in a poor family. In 1807 Lobachevski graduated from the Gymnasium and entered Kazan University. Lobachevski was highly successful in all courses he took. Martin Bartels (1769 - 1833) was Professor of Mathematics. Bartels was a school teacher and friend of Gauss. Gauss may have given Lobachevski mathematical hints through the letters between Bartels and Gauss. Lobachevski received a Master's Degree in physics and mathematics in 1811. In 1814 he was appointed to a lectureship and in 1816 he became an extraordinary professor.

In 1822 he was appointed as a full professor . . . the same year in which he began an administrative career as a member of the committee formed to supervise the construction of new university buildings. He was appointed to important positions within the university such as the Dean of the Mathematics and Physics Department between 1820 and 1825 and head librarian from 1825 to 1835. He also served as Head of the Observatory.

In 1827 Lobachevski became rector of Kazan University, a post he was to hold for the next 19 years. Two natural disasters struck the university while he was Rector of Kazan . . . a cholera epidemic in 1830 and a big fire in 1842. Owing to resolute and reasonable measures taken by Lobachevski the damage to the University was reduced to a minimum. Lobachevski continued to teach a variety of different topics such as mechanics, hydrodynamics, integration, differential equations, the calculus of variations, and mathematical physics.

After Lobachevski retired in 1846, his health rapidly deteriorated. The illnes became progressively worse and led to blindness. These and financial difficulties added to the heavy burdens he had to bear over his last years. His great mathematical achievements were not recognised in his lifetime.

Lobachevski's major work, *Geometriya*, completed in 1823, was not published in its original form until 1909. On 11 February 1826, in the session of the Department of Physico-Mathematical Sciences at Kazan University, Lobachevski requested that his work about a new geometry be heard.

In 1837 Lobachevski published his article "Géométrie imaginaire" and a summary of his new geometry *Geometrische Untersuchungen zur Theorie der Parellellinien* was published in Berlin in 1840. This last publication greatly impressed Gauss.

There were two further major contributions to Lobachevski's geometry by Poincaré in 1882 and 1887. Perhaps these finally mark the acceptance of Lobachevski's ideas which would eventually be seen as vital steps in freeing the thinking of mathematicians so that relativity theory had a natural mathematical foundation.

The idea of constructing a *model* is the key to verifying consistency. When you write down a collection of axioms, you are describing a mathematical system or object that may or may not exist. But if you can provide an *independent construction* of the alleged object, one that satisfies the axioms, then you have explicitly exhibited an entity that fits the specified description. The axioms cannot lead to a contradiction because the model you have constructed shows that there are objects that possess the specified properties. Indeed, one of Kurt Gödel's fundamental theorems guarantees that any axiomatic system for which there is a model will never lead to a contradiction. [Again, this is a slight oversimplification; for all mathematical notions of consistency are *relative notions*. You will learn more about this idea in a course on formal logic.]

Model theory is an advanced and subtle idea. After you take an advanced course in logic, you may wish to refer to [CHK] for more on this topic.