Math 310
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Figure: This is your instructor.

The last lecture was rather long, so this one will be on the short side.

We have already seen that the operation of set-theoretic product is analogous to multiplication in ordinary arithmetic. For the cardinality of $S \times T$ is the product of the cardinalities of S and T when the latter two are finite.

The other arithmetic operations may be modeled by set-theoretic operations. In this section we treat a few of these.

Definition: Let S and T be sets. Define T^S to be the set of all functions from S to T.

Example: Let $S = \{a, b, c\}$ and $T = \{1, 2\}$. The members of T^S are

$$\begin{aligned}
f_1 &= \{(a, 1), (b, 1), (c, 1)\} \\
f_2 &= \{(a, 1), (b, 1), (c, 2)\} \\
f_3 &= \{(a, 1), (b, 2), (c, 1)\} \\
f_4 &= \{(a, 1), (b, 2), (c, 2)\} \\
f_5 &= \{(a, 2), (b, 1), (c, 1)\} \\
f_6 &= \{(a, 2), (b, 1), (c, 2)\} \\
f_7 &= \{(a, 2), (b, 2), (c, 1)\} \\
f_8 &= \{(a, 2), (b, 2), (c, 2)\}
\end{aligned}$$

Notice in the last example that there are $\mathbf{8} = \mathbf{2}^{\mathbf{3}}$ elements of T^S, which is precisely equal to $(\mathrm{card}\,\mathrm{T})^{\mathrm{card}\,\mathrm{S}}$. This is an instance of a general phenomenon:

Proposition: *Let* S *and* T *each be sets with finite cardinality. Then*

$$\mathrm{card}\big(\mathrm{T}^{\mathrm{S}}\big) = (\mathrm{card}\,\mathrm{T})^{\mathrm{card}\,\mathrm{S}}.$$

Proof: Let the cardinality of T be k and the cardinality of S be m. Write $T = \{t_1, \ldots, t_k\}$ and $S = \{s_1, \ldots, s_m\}$. If f is a function from S to T, then there are k possible values that $f(s_1)$ may take. Likewise, there are k possible values that $f(s_2)$ may take. Repeating this analysis a total of m times, we find that there are

$$\underbrace{k \cdot k \cdots k}_{m \text{ times}} = k^m$$

possible functions from S to T. □

Let us briefly treat the operation of set-theoretic product. If S and T are sets, then we have learned that $S \times T$ is the set of ordered pairs (s, t) such that $s \in S$ and $t \in T$. Once we note that $(S \times T) \times U$ can be identified with $S \times (T \times U)$ in a natural way, then we may meaningfully consider a *finite* product $S_1 \times S_2 \times \cdots \times S_k$. This is defined to be the set of all ordered k-tuples $(s_1, s_2, \ldots, s_k)$ such that $s_j \in S_j$ for $j = 1, \ldots, k$. As a simple example, Euclidean space $\mathbb{R}^3$ is nothing other than $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

But now consider sets S_α for α in some possibly large (even infinite) index set A. We wish to define

$$\prod_{\alpha \in A} S_\alpha .$$

If A is an uncountable set, then it is essentially impossible to speak of ordered -tuples with the same cardinality as A. Yet it is frequently useful in mathematics to be able to treat this notion.

We motivate our new definition by recasting the product of two sets in a new language. Let S_1, S_2 be sets. Let us define $S_1 \times S_2$ to be the set of all functions $f : \{1, 2\} \to S_1 \cup S_2$ such that $f(1) \in S_1$ and $f(2) \in S_2$. Notice that each such function f can be thought of as the ordered pair $(f(1), f(2))$, with the first entry in S_1 and the second entry in S_2. So the language of functions allows us to express ordered pairs in a new way. Likewise, if T_1, T_2, T_3 are sets then we may define $T_1 \times T_2 \times T_3$ to be the set of all functions $f : \{1, 2, 3\} \to S_1 \cup S_2 \cup S_3$ such that $f(j) \in S_j, j = 1, 2, 3$. A moment's thought shows that this new definition is consistent with our notion of the product of three sets as discussed above. We may construct a similar definition for the product of any k sets, k a positive integer.

Now return to the consideration of a collection $\{S_\alpha\}_{\alpha \in A}$ of (possibly uncountably many) sets, indexed over a set A. Then the *product* of these sets is defined to be the set of all functions $f : A \to \left[\cup_{\alpha \in A} S_\alpha\right]$ such that $f(\alpha) \in S_\alpha$. We are no longer at liberty to compare this new definition with some more concrete set of ideas. For a large index set A, this must stand as our definition of product.

Example: For each $\alpha \in A = \mathbb{R}$, let $S_\alpha = \mathbb{Z}$. Then $\prod_{\alpha \in A} S_\alpha$ is just the collection of all functions from $\mathbb{R}$ into $\mathbb{Z}$ or, in the language introduced at the beginning of this section, $\mathbb{Z}^{\mathbb{R}}$. Of course this set has cardinality strictly greater than the cardinality of the continuum.