Math 310
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Figure: This is your instructor.

The Natural Number System

Giuseppe Peano's axioms for the natural numbers are as follows. In this discussion, we will follow tradition and use the notation $'$ to denote the "successor" of a natural number. For instance, the successor of 2 is $2'$. Intuitively, the successor of n is the number $n + 1$. However addition is something that comes later; so we formulate the basic properties of the natural numbers in terms of the successor function.

PEANO'S AXIOMS FOR THE NATURAL NUMBERS

P1 $1 \in \mathbb{N}$.

P2 If $n \in \mathbb{N}$, then $n' \in \mathbb{N}$.

P3 There is no natural number n such that $n' = 1$.

P4 If m and n are natural numbers and if $m' = n'$, then $m = n$.

P5 Let P be a property. If

(i) $P(1)$ is true;

(ii) $P(j) \Rightarrow P(j')$ for every $j \in \mathbb{N}$

then $P(n)$ is true for every $n \in \mathbb{N}$.

As Suppes says in [SUP, pp. 121 ff.], these axioms for the natural numbers are almost universally accepted (although E. Nelson [NEL], among others, has found it useful to explore how to develop arithmetic without Axiom **P5**). They have evolved into their present form so that the natural numbers will satisfy those properties that are generally recognized as desirable. Let us briefly mention what each of the axioms signifies:

- P1 asserts that $\mathbb{N}$ contains a distinguished element that we denote by 1.
- P2 asserts that each element of $\mathbb{N}$ has a successor.
- P3 asserts that 1 is not the successor of any natural number; in other words, 1 is in a sense the "first" element of $\mathbb{N}$.
- P4 asserts that if two natural numbers have the same successor then they are in fact the same natural number.
- P5 asserts that the method of mathematical induction is valid.

Some obvious, and heuristically appealing, properties of the natural numbers can be derived rather directly from Peano's axioms. Here is an example:

Proposition *Let n be a natural number other than 1. Then $n = m'$ for some natural number m.*

Remark: This proposition makes the intuitively evident assertion that every natural number, except 1, is a successor. Another way of saying this is that every natural number except 1 has a predecessor. This claim is clearly not an explicit part of any of the five axioms. In fact the only axiom that has a hope of implying the proposition is the inductive axiom, as we shall now see.

Proof: Let $P(j)$ be the statement "either $j = 1$ or $j = n'$ for some natural number n."

Clearly $P(j)$ is true when $j = 1$.

Now suppose that the statement $P(j)$ has been established for some natural number j. We wish to establish it for j'. But j' is, by definition, a successor. So the statement is true.

This completes our mathematical induction. $\square$

This proof is misleading in its simplicity. The proof consists of little more than *interpreting* axiom **P5**. Some other desirable properties of the natural numbers are much more difficult to prove directly from the axioms. As an instance, to prove that there is no natural number lying between k and k' is complicated. Indeed, the entire concept of ordering is extremely tricky. And the single most important property of the natural numbers (one that is essentially equivalent to the mathematical induction axiom, as we shall later see) is that the natural numbers are well ordered in a canonical fashion. So we must find an efficient method for establishing the properties of the natural numbers that are connected with order.

It is generally agreed (see [SUP, p. 121 ff.]) that the best way to develop further properties of the natural numbers is to treat a specific model. Even that approach is nontrivial; so we shall only briefly sketch the construction of a model and further sketch its order properties so that we may discuss well ordering. The approach that we take is by way of the so-called *finite ordinal arithmetic*.

Let us define a model for the natural numbers as follows:

$\mathbf{1} = \{\emptyset\}$
$\mathbf{2} = \mathbf{1} \cup \{\mathbf{1}\} = \{\emptyset, \{\emptyset\}\}$
$\mathbf{3} = \mathbf{2} \cup \{\mathbf{2}\} = \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$

and, in general,

$k' = k \cup \{k\}$.

It is straightforward to verify Peano's axioms for this model of the natural numbers. Let us first notice that, in this model, the successor n' of a natural number n is given by $n' = n \cup \{n\}$. Now let us sketch the verification of the axioms.

P1 is clear by construction, and so is **P2**.

P3 is an amusing exercise in logic: If $m' = \mathbf{1}$, then there is a set A such that $A \cup \{A\} = \mathbf{1}$ or $A \cup \{A\} = \{\emptyset\}$. In particular, $x \in A \cup \{A\}$ implies $x \in \{\emptyset\}$. Since $A \in A \cup \{A\}$, it follows that $A = \emptyset$. But $\emptyset$ is not a natural number. So $\mathbf{1}$ is not the successor of a natural number.

For **P4**, it is convenient to invoke the concept of ordering in our model of the natural numbers. Say that $m < n$ if $m \in n$. Clearly, if $m, n \in \mathbb{N}$ and $m \neq n$, then either $m \in n$ or $n \in m$ but not both. Thus we have the usual trichotomy of a strict, simple order. Now suppose that m, n are natural numbers and that $m' = n'$. If $m < n$, then $m \in n$ so $m' \in n'$ and $m' < n'$. That is false. Likewise we cannot have $n < m$. Thus it must be that $m = n$.

We shall discuss **P5** a bit later.

Next we turn to well ordering. We assert that our model of the natural numbers, with the ordering defined in our discussion of **P4**, has the property that if $\emptyset \neq S \subset \mathbb{N}$, then there is an element $s \in S$ such that $s < t$ for every $s \neq t \in S$. It is clear from the trichotomy that the least element s, if it exists, is unique. We proceed in several steps:

Fix a natural number $m > \mathbf{1}$ and restrict attention to $\mathcal{Q}(m)$, which is the set of natural numbers that are less than m.

Proposition *The set $\mathcal{Q}(m)$ has finitely many elements.*

Proof: *A natural number k is less than m if and only if $k \in m$. But, by construction, m has only finitely many elements.* $\square$

Remark: As an exercise, you may wish to attempt to prove this last proposition directly from the Peano axioms.

Proposition *For each m, the set $\mathcal{Q}(m)$ is well ordered.*

Proof: The proof is by mathematical induction on m.

When $m = \mathbf{1}$, there is nothing to prove.

Assume that the assertion has been proved for $m = j$. Now let U be a subset of $\mathcal{Q}(j')$. There are now three possibilities:

(1) If in fact $U \subset \mathcal{Q}(j)$, then U has a least element by the inductive hypothesis.

(2) If $U = \{j\}$, then U has but one element and that element, namely j, is the minimum that we seek.

(3) The last possibility is that U contains j and some other natural numbers as well. But then $U \setminus \{j\} \subset \mathcal{Q}(j)$. Hence $U \setminus \{j\}$ has a least element s by the inductive hypothesis. Since s is automatically less than j, it follows that s is a least element for the entire set U. $\square$

Now we have all our tools in place, and we can prove the full result:

Theorem: *The natural numbers* $\mathbb{N}$ *are well ordered.*

Proof: Let $\emptyset \neq S \subset \mathbb{N}$. Select an element $m \in S$. There are now two possibilities:

(1) If $\mathcal{Q}(m) \cap S = \emptyset$, then m is the least element of S that we seek.

(2) If $T = \mathcal{Q}(m) \cap S \neq \emptyset$, then notice that, if $x \in T$ and $y \in S \setminus T$, then $x < y$. So it suffices for us to find a least element of T. But such an element exists by the preceding proposition.

The proof is complete. $\square$

We next observe that, in a certain sense, the well-ordering property implies the mathematical induction property (Axiom **P5**). By this we mean the following: Do not consider any model of the natural numbers, but just consider any number system X satisfying **P1** - **P4**. Assume that every element of X except **1** has a predecessor, and in addition assume that this number system is well ordered.

Now let P be a property. Assume that $P(\mathbf{1})$ is true, and assume the syllogism $P(j) \Rightarrow P(j')$ for all j. We claim that $P(n)$ is true for every n. Suppose not. Then the set

$$S = \{m \in X : P(m) \text{ is false}\}$$

is nonempty. Let q be the least element of S; this number is guaranteed to exist by the well-ordering property. The number q cannot be 1, for we assumed that $P(\mathbf{1})$ is true. But if $q > \mathbf{1}$, then q has a predecessor r. Since q is the least element of S, then it cannot be that $r \in S$. Thus $P(r)$ must be true. But then, by our hypothesis, $P(r')$ must be true. However, $r' = q$. So $P(q)$ is true. That is a contradiction.

The only possible conclusion is that S is empty. So $P(n)$ is true for every n.

These remarks about well-ordering implying **P5** are not satisfactory from our point of view, because *we used the inductive property* to establish that every natural number other than 1 has a predecessor. But it is important for you to understand that any development of the natural numbers will result in mathematical induction and well ordering being closely linked.

If you review your calculus book or other elementary texts, you will find that both mathematical induction and well ordering are occasionally used. But in every instance the author will say "These properties of the natural numbers are intuitively clear; trust me." Now you can begin to understand why an elementary textbook author must make that choice. The truth about these topics is inexorably linked to the very foundations of mathematics, and is therefore both subtle and complicated.

The remaining big idea connected with the natural numbers is addition. It can be proven that a satisfactory theory of addition *cannot* be developed from **P1** - **P5** (see [SUP, p. 136] for details). Instead, it is customary to adjoin two new axioms to our theory:

P6 If x is a natural number, then $x + \mathbf{1} = x'$;

P7 For any natural numbers x, y we have

$$x + y' = (x + y)'.$$

To illustrate these ideas, let us close the section by proving that $\mathbf{2} + \mathbf{2} = \mathbf{4}$. [Judge for yourself whether the proof is as obvious as $\mathbf{2} + \mathbf{2} = \mathbf{4}$!]. In this argument we use the definitions $\mathbf{2} = \mathbf{1}'$, $\mathbf{3} = \mathbf{2}'$, $\mathbf{4} = \mathbf{3}'$.

$$\begin{aligned}
\mathbf{2} + \mathbf{2} &= \mathbf{2} + \mathbf{1}' && \text{by definition of } \mathbf{2} \\
&= (\mathbf{2} + \mathbf{1})' && \text{by } \mathbf{P7} \\
&= (\mathbf{2}')' && \text{by } \mathbf{P6} \\
&= \mathbf{3}' && \text{by definition of } \mathbf{3} \\
&= \mathbf{4} && \text{by definition of } \mathbf{4}
\end{aligned}$$

In fact, it is even trickier to get multiplication to work in the natural number system. Almost the only viable method is to add even more axioms that control this binary operation. We can say no more about the matter here, but refer to [SUP].

It is not difficult to see that, with enough patience (or with mathematical induction), one could establish all the basic laws of arithmetic. Of course this would not be a fruitful use of our time. The celebrated work [WHB] treats this matter in complete detail.

In the succeeding sections of the present book, we shall take the basic laws of arithmetic on the natural numbers as given. We understand that our treatment of the natural numbers is incomplete. We have touched on some topics, and indicated some constructions. But, when it comes right down to it, we are taking the natural numbers on faith. All of our future number systems (the integers, the rational numbers, the reals, the complexes, the quaternions) will be constructed rigorously. The somewhat bewildering situation before us is that the more complicated numbers systems are easier to construct.