Math 310
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Figure: This is your instructor.

More on the Integers

Remark What is the point of the last lecture? Everyone knows about negative numbers, so why go through this abstract construction? The reason is that, until one sees this construction, negative numbers are just imaginary objects—placeholders if you will—which are a useful notation but which do not exist. Now they *do* exist. They are a collection of equivalence classes of pairs of natural numbers. This collection is equipped with certain arithmetic operations, such as addition, subtraction, and multiplication. We now discuss these last two.

If $A = [(a, b)]$ and $C = [(c, d)]$ are integers, then we define their *difference* to be the equivalence class $[(a + d, b + c)]$; we denote this difference by $A - C$. [Note that we may not use subtraction of natural numbers in our definition of subtraction of integers; subtraction of natural numbers is not, in general, defined.] The unambiguity (or well definedness) of this definition is treated in the exercises.

Example We calculate $8 - 14$. Now $8 = [(1, 9)]$ and $14 = [(3, 17)]$. Therefore

$$8 - 14 = [(1 + 17, 9 + 3)] = [(18, 12)] = -6,$$

as expected.

As a second example, we compute $(-4) - (-8)$. Now

$$-4 - (-8) = [(6, 2)] - [(13, 5)] = [(6 + 5, 2 + 13)] = [(11, 15)] = 4.$$

Remark When we first learn that $(-4) - (-8) = (-4) + 8 = 4$, the explanation is a bit mysterious: why is "minus a minus equal to a plus"? Now there is no longer any mystery: this property follows *from our construction* of the number system $\mathbb{Z}$.

Remark It is interesting to sort out the last example from the justification for the arithmetic of negative numbers that we learn in high school. Here is an example of that reasoning.

It is postulated that negative numbers exist (they certainly are not constructed). Then it is noted that

$$18 + (8 - 14) = (18 - 14) + 8 = 4 + 8 = 12 = 18 - 6 = 18 + (-6) .$$

Identifying the far left and far right sides of the equation, we cancel 18 from each side and conclude that $8 - 14 = -6$.

This reasoning is perfectly correct. But it presupposes the existence of a number system that **(i)** contains negative integers and **(ii)** obeys all the familiar laws of arithmetic.

The advantage of the presentation in this and the last lecture is that we actually *construct* such a number system. We do not presuppose it. The additive properties of negative numbers follow automatically from our construction. They are not derived by algebraic tricks from some numbers that we do not actually know exist.

Finally, we turn to multiplication. If $A = [(a, b)]$ and $C = [(c, d)]$ are integers, then we define their product by the formula

$$A \cdot C = [(a \cdot d + b \cdot c, a \cdot c + b \cdot d)].$$

This definition may be a surprise. Why did we not define $A \cdot C$ to be $[(a \cdot c, b \cdot d)]$? There are several reasons: first of all, the latter definition would give the wrong answer; moreover, it is not unambiguous (different representatives of A and C would give a different answer). If you recall that we think of $[(a, b)]$ as representing $b - a$ and $[(c, d)]$ as representing $d - c$, then the product should be the equivalence class that represents $(b - a) \cdot (d - c)$. That is the motivation behind our definition.

The unambiguity (or well definedness) of the given definition of multiplication of integers is treated in the exercises. We proceed now to an example.

Example We compute the product of -3 and -6. Now

$$(-3)\cdot(-6) = [(5,2)]\cdot[(9,3)] = [(5\cdot3+2\cdot9, 5\cdot9+2\cdot3)] = [(33,51)] = 18,$$

which is the expected answer.

As a second example, we multiply -5 and 12. We have

$$-5\cdot12 = [(7,2)]\cdot[(1,13)] = [(7\cdot13+2\cdot1, 7\cdot1+2\cdot13)] = [(93,33)] = -60.$$

Finally, we show that 0 times any integer A equals 0. Let $A = [(a, b)]$. Then

$$0 \cdot A = [(1,1)] \cdot [(a,b)] = [(1 \cdot b + 1 \cdot a, 1 \cdot a + 1 \cdot b)] = [(a+b, a+b)] = 0.$$

Remark Notice that one of the pleasant by-products of our construction of the integers is that we no longer have to give artificial explanations for why the product of two negative numbers is a positive number or why the product of a negative number and a positive number is negative. These properties instead follow *automatically* from our construction.

Remark It is interesting to sort out the last example from the justification for the arithmetic of negative numbers that we learn in high school. Here is an example of that reasoning.

It is postulated that negative numbers exist (they certainly are not constructed). Then it is noted that

$$3 \cdot 8 = (6 - 3) \cdot 8 = 6 \cdot 8 - 3 \cdot 8$$

hence

$$24 = 48 - 3 \cdot 8$$

or, using reasoning as in our last remark but one,

$$-24 = -3 \cdot 8 \,.$$

Similarly, one can show that

$$-48 = -6 \cdot 8 \,.$$

Taking these two facts for granted, we then compute that

$$(8-3)\cdot(8-6) = 8\cdot 8 + 8\cdot(-6) + (-3)\cdot 8 + (-3)\cdot(-6)\,.$$

As a result,

$$10 = 64 - 48 - 24 + (-3)\cdot(-6)$$

or

$$10 + 72 - 64 = (-3)\cdot(-6)\,,$$

hence

$$18 = (-3)\cdot(-6)\,.$$

Again, this reasoning is perfectly correct. But it presupposes the existence of a number system that **(i)** contains negative integers and **(ii)** obeys all the familiar laws of arithmetic.

The advantage of the presentation in this section of the present book is that we actually *construct* such a number system. We do not presuppose it. The multiplicative properties of negative numbers follow automatically from our construction. They are not derived by algebraic tricks from some numbers that we do not actually know exist.

Notice that the integers $\mathbb{Z}$ as we have constructed them contain the element $0 \equiv [(1,1)]$. This element is the *additive identity* in the sense that $x + 0 = 0 + x = x$ for any integer x. Also, if $y = [(a,b)]$ is any integer, then it has an *additive inverse* $-y = [(b,a)]$. This means that $y + (-y) = 0$. As a result of these two facts, the integers $\mathbb{Z}$ form a *group*. We shall say more about groups at the end of the course.

The integer system that we have constructed also contains a multiplicative identity. It is $1 = [(1,2)]$. In fact if $A = [(a,b)]$ is any integer, then $1 \cdot A = A \cdot 1 = A$. To see this, let us calculate $1 \cdot A$.

$$1 \cdot A = (1,2) \cdot (a,b) = (1 \cdot b + 2 \cdot a, 1 \cdot a + 2 \cdot b) = (b + 2a, a + 2b) .$$

This does not look like (a,b). But it is related to (a,b) because $a + (a + 2b) = b + (b + 2a)$.

Of course we will not discuss division for integers; in general division of one integer by another makes no sense *in the universe of the integers*. More will be said about this fact in the exercises.

In the rest of this class, we shall follow the standard mathematical custom of denoting the set of all integers by the symbol $\mathbb{Z}$. We will write the integers not as equivalence classes, but in the usual way as the sequence of digits $\cdots - 3, -2, -1, 0, 1, 2, 3, \ldots$. The equivalence classes are a device that we used to *construct* the integers. Now that we have them, we may as well write them in the simple, familiar fashion and manipulate them as usual.

In an exhaustive treatment of the construction of $\mathbb{Z}$, we would prove that addition and multiplication are commutative and associative, prove the distributive law, and so forth. But the purpose of this section is to demonstrate modes of logical thought rather than to be exhaustive. We shall say more about some of the elementary properties of the integers in the exercises.