Math 310
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Figure: This is your instructor.

The Polar Form of a Complex Number

Let θ be any real number. A famous formula of Euler asserts that

$$e^{i\theta} = \cos\theta + i\sin\theta .$$

A rigorous verification of this formula requires a study of complex power series. We now provide you with an intuitive argument that should make you comfortable with Euler's formula.

If z is *any* complex number, then define

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}.$$

Notice that, when z happens to be a real number, then the formula is one that you learned in calculus. The new formula is a standard generalization of the calculus formula. Substitute in $i\theta$ for z and (manipulating the series just as though it were a polynomial) separate the right-hand side into its real and imaginary parts.

The result is

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \cdots\right).$$

Finally, notice that the power series expansions in the parentheses on the right are those associated with the functions cosine and sine, respectively. Thus

$$e^{i\theta} = \cos\theta + i\sin\theta\,.$$

This is Euler's formula.

If $\xi = s + it$ is any complex number such that $s^2 + t^2 = 1$, then we may find an angle θ, $0 \leq \theta < 2\pi$, such that $\cos\theta = s$ and $\sin\theta = t$. See the next figure. We conclude that

$$\xi = e^{i\theta}.$$

Explain this reasoning in detail.

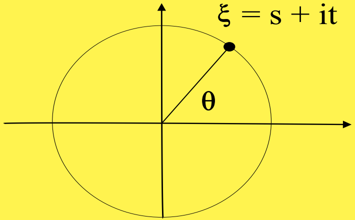

Figure: The angle associated to a complex number of modulus 1.

If $z = x + iy \in \mathbb{C}$ is any nonzero complex number, then let

$$r^2 = |z|^2 = x^2 + y^2 .$$

The number r is the distance of z to the origin in the Argand plane. It is also the modulus of z. Set $\xi = z/r$. Show that $|\xi| = 1$. Now apply the result from the preceding frame to conclude that

$$z = r \cdot \xi = r\, e^{i\theta} ,$$

some $0 \leq \theta < 2\pi$. This is called the *polar form* of the complex number z.

If $z = re^{i\theta}$ is a complex number in polar form, then $re^{i(\theta + 2\pi)}$ is the same complex number. This statement follows from the periodicity of sine and cosine.

Let us use these ideas to find all cube roots of i. Using the polar notation, we can write

$$i = 1 \cdot e^{i(\pi/2)} \,.$$

We need to solve the equation

$$(re^{i\theta})^3 = i = 1 \cdot e^{i(\pi/2)}$$

or

$$r^3 \cdot e^{3i\theta} = 1 \cdot e^{i(\pi/2)} \,.$$

We see that

$$r^3 = 1 \qquad \text{and} \qquad 3i\theta = i\pi/2 \,.$$

In conclusion
$$r = 1 \qquad \text{and} \qquad \theta = \frac{\pi}{6} \, .$$
Thus we have found that one cube root of i is
$$z_1 = 1 \cdot e^{i\pi/6} = \cos(\pi/6) + i \sin(\pi/6) = \frac{\sqrt{3}}{2} + i \cdot \frac{1}{2} \, .$$
We are not finished because we expect a nonzero complex number α to have three cube roots (after all, these are the roots of the equation $z^3 - i = 0$).

So now we look at the equation

$$(re^{i\theta})^3 = 1 \cdot e^{i(\pi/2+2\pi)} .$$

Here of course we are using the periodicity of sine and cosine. Now we have

$$r^3 = 1 \qquad \text{and} \qquad 3\theta = \frac{5\pi}{2} .$$

We find that

$$r = 1 \qquad \text{and} \qquad \theta = \frac{5\pi}{6} .$$

In conclusion, we have a second cube root

$$z_2 = 1 \cdot e^{(5\pi/6)i} = \cos\frac{5\pi}{6} + i\sin\frac{5pi}{6} = -\frac{\sqrt{3}}{2} + i\frac{1}{2} .$$

Repeating this process once more, we have

$$(re^{i\theta})^3 = 1 \cdot e^{i(\pi/2+4\pi)} .$$

Here of course we are using the periodicity of sine and cosine. Now we have

$$r^3 = 1 \qquad \text{and} \qquad 3\theta = \frac{9\pi}{2} .$$

We find that

$$r = 1 \qquad \text{and} \qquad \theta = \frac{3\pi}{2} .$$

In conclusion, we have a second cube root

$$z_3 = 1 \cdot e^{(3\pi/2)i} = \cos\frac{3\pi}{2} + i\sin\frac{3pi}{2} = 0 - i = -i .$$

That is the complete solution to the problem of finding all cube roots of i.

Let us do one more example of this type. We will find all fourth roots of -16. Now

$$-16 = 16 \cdot e^{i\pi} .$$

So we must solve

$$(re^{i\theta})^4 = 16e^{i\pi} .$$

It follows that

$$r^4 = 16 \qquad \text{and} \qquad 4\theta = \pi .$$

The result is that

$$r = 2 \qquad \text{and} \qquad \theta = \frac{\pi}{4}\,.$$

So one fourth root of -16 is

$$z_1 = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2} + i\sqrt{2}\,.$$

Now let us pass to the next fourth root. We must solve

$$(re^{i\theta})^4 = 16e^{i(\pi+2\pi)} .$$

It follows that

$$r^4 = 16 \qquad \text{and} \qquad 4\theta = 3\pi .$$

The result is that

$$r = 2 \qquad \text{and} \qquad \theta = \frac{3\pi}{4} .$$

So the second fourth root of -16 is

$$z_2 = 2e^{i3\pi/4} = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} + i\sqrt{2}$$

Next let us pass to the third fourth root. We must solve

$$(re^{i\theta})^4 = 16e^{i(\pi+4\pi)} .$$

It follows that

$$r^4 = 16 \qquad \text{and} \qquad 4\theta = 5\pi .$$

The result is that

$$r = 2 \qquad \text{and} \qquad \theta = \frac{5\pi}{4} .$$

So the third fourth root of -16 is

$$z_3 = 2e^{i5\pi/4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\sqrt{2} - i\sqrt{2}$$

Finally let us pass to the last fourth root. We must solve

$$(re^{i\theta})^4 = 16e^{i(\pi+6\pi)}\,.$$

It follows that

$$r^4 = 16 \qquad \text{and} \qquad 4\theta = 7\pi\,.$$

The result is that

$$r = 2 \qquad \text{and} \qquad \theta = \frac{7\pi}{4}\,.$$

So the last fourth root of -16 is

$$z_4 = 2e^{i7\pi/4} = 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = \sqrt{2} - i\sqrt{2}\,.$$

Now we have found all fourth roots of -16.

It is interesting to note that, if you plot the cube roots of i as ordered pairs in the plane (this is called an *Argand diagram*), then you will see that the three roots are equally spaced around a circle of radius 1 centered at the origin.

If instead you plot the fourth roots of -16 as ordered pairs in the plane, then you will see that the four roots are equally spaced around a circle of radius 2 centered at the origin.

As an exercise, you may wish to try your hand at calculating the fourth roots of $-2i$. Plot your four roots in an Argand diagram, and observe that they are equally spaced around a circle centered at the origin.