Math 310
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Figure: This is your instructor.

The Quaternions and the Cayley Numbers

Now we shall discuss a number system that you may have never encountered before. It is called the system of *quaternions*. Our description will be an informal one.

The quaternions were discovered by William Rowan Hamilton (1805–1865). He had been trying for many years to produce a field structure on $\mathbb{R}^3$. One day, walking to an important meeting with his wife, he realized that he should not be wedded to commutativity of multiplication, and also not be wedded to dimension 3. He was so excited by his discovery that he whipped out his penknife and carved the result onto the bridge that they were crossing. So now we have the quaternions. You might think it natural to denote the quaternions by $\mathbb{Q}$. But that notation is already reserved for the rational numbers. Instead we denote the quaternions by $\mathbb{H}$, in honor of Hamilton.

Imagine $\mathbb{R}^4 \equiv \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ equipped with the following operations: set $\mathbf{i} = (0,1,0,0)$, $\mathbf{j} = (0,0,1,0)$, $\mathbf{k} = (0,0,0,1)$. Denote the 4-tuple (1,0,0,0) by $\mathbf{1}$. Define the multiplication laws

$$\mathbf{i} \cdot \mathbf{i} = -\mathbf{1} \quad , \quad \mathbf{j} \cdot \mathbf{j} = -\mathbf{1} \quad , \quad \mathbf{k} \cdot \mathbf{k} = -\mathbf{1}$$

and

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{k} \quad , \quad \mathbf{j} \cdot \mathbf{k} = \mathbf{i} \quad , \quad \mathbf{k} \cdot \mathbf{i} = \mathbf{j}$$

and

$$\mathbf{j} \cdot \mathbf{i} = -\mathbf{k} \quad , \quad \mathbf{k} \cdot \mathbf{j} = -\mathbf{i} \quad , \quad \mathbf{i} \cdot \mathbf{k} = -\mathbf{j}.$$

Of course the element $\mathbf{1}$ multiplied times any 4-tuple z is declared to be equal to z. In particular, $\mathbf{1} \cdot \mathbf{1} = \mathbf{1}$.

Finally, if $z = (z_1, z_2, z_3, z_4)$ and $w = (w_1, w_2, w_3, w_4)$ are 4-tuples, then we write

$$z = z_1 \cdot \mathbf{1} + z_2\mathbf{i} + z_3\mathbf{j} + z_4\mathbf{k}$$

and

$$w = w_1 \cdot \mathbf{1} + w_2\mathbf{i} + w_3\mathbf{j} + w_4\mathbf{k}.$$

Then $z \cdot w$ is defined by using the (obvious) distributive law and the rules already specified.

Example:

$$
\begin{aligned}
(2,0,1,3) \cdot (\text{-}4,1,0,1) &= [2 \cdot \mathbf{1} + \mathbf{j} + 3\mathbf{k}] \cdot [-4 \cdot \mathbf{1} + \mathbf{i} + \mathbf{k}] \\
&= (2 \cdot (-4)) \cdot \mathbf{1} + (2\mathbf{i}) + (2\mathbf{k}) \\
&\quad +(\mathbf{j} \cdot (-\mathbf{4})) + (\mathbf{j} \cdot \mathbf{i}) + (\mathbf{j} \cdot \mathbf{k}) \\
&\quad +(3\mathbf{k} \cdot (-4)) + (3\mathbf{k} \cdot \mathbf{i}) + (3\mathbf{k} \cdot \mathbf{k}) \\
&= -8 \cdot \mathbf{1} + 2\mathbf{i} + 2\mathbf{k} - 4\mathbf{j} - \mathbf{k} + \mathbf{i} \\
&\quad -12\mathbf{k} + 3\mathbf{j} - 3 \cdot \mathbf{1} \\
&= -11 \cdot \mathbf{1} + 3\mathbf{i} - \mathbf{j} - 11\mathbf{k} \\
&= (-11, 3, -1, -11).
\end{aligned}
$$

Addition of two quaternions is simply performed componentwise: if $z = (z_1, z_2, z_3, z_4)$ and $w = (w_1, w_2, w_3, w_4)$, then

$$z + w = (z_1 + w_1, z_2 + w_2, z_3 + w_3, z_4 + w_4).$$

Verify for yourself that the additive identity in the quaternions is $\mathbf{0} = (0,0,0,0)$. The multiplicative identity is $\mathbf{1} = (1,0,0,0)$.

In fact it can be checked that each nonzero element of the quaternions has a unique two-sided multiplicative inverse. However, since multiplication is not commutative, the quaternions do not form a field; instead the algebraic structure is called a *division ring*.

It is also possible to give $\mathbb{R}^8$ an additive and a multiplicative structure. The multiplication operation is both noncommutative and nonassociative. The resulting eight dimensional algebraic object is called the *Cayley numbers* or *octonions*. We shall not present the details here. It is one of the great theorems of twentieth century mathematics (see [ADA], [BOM]) that $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^4$, and $\mathbb{R}^8$ are the only Euclidean spaces that can be equipped with compatible addition and multiplication operations in a natural way (so that the algebraic operations are smooth functions of the coordinates).

The quaternions and Cayley numbers are used in mathematical physics, in the representation theory of groups, and in algebraic topology. Every cell phone uses the Cayley numbers for coding.

More on the Real Numbers

The real numbers are a profound and complex world. We earlier had an introduction to the real numbers, but we did not explore any of their truly deep properties.

In this current brief discussion we begin to explore the real numbers and establish some of their more remarkable aspects. This will be a real mathematical adventure, and you should prepare to enjoy it.

If x is a real number, then the *absolute value* of x, denoted $|x|$, is the distance of x to 0. In other words,

$$|x| = \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -x & \text{if} \quad x < 0 \,. \end{cases}$$

Fundamental to our study of the deeper properties of the real numbers is the *triangle inequality*: If x, y are real numbers, then

$$|x + y| \leq |x| + |y| . \qquad (*)$$

In fact the standard triangle inequality $(*)$ entails other inequalities that are also useful. Let $x = a + b$ and $y = -b$. Then $(*)$ implies

$$|(a + b) - b| \leq |a + b| + |b|$$

hence

$$|a| - |b| \leq |a + b| . \qquad (**)$$

A *sequence* in $\mathbb{R}$ is a function $\phi : \mathbb{N} \to \mathbb{R}$. We denote the elements of the sequence by $\phi(1), \phi(2), \ldots$. For example,

$$\phi(j) = j^2 + 1$$

is a sequence. It is often useful to write out the elements of the sequence in order: $2, 5, 10, 17, \ldots$. We frequently denote the elements of a sequence by the more convenient notation $\phi_1, \phi_2, \phi_3, \ldots$ (rather than think of the sequence as a function). The principal property of a sequence is whether or not it *converges*.

We say that a sequence $\{a_j\} = \{a_1, a_2, \ldots\}$ *converges* to a number α if, for every $\epsilon > 0$, there is a positive integer K such that $j > K$ implies that $|a_j - \alpha| < \epsilon$. What we have enunciated is a quantitative, rigorous way of asserting that the numbers a_j become closer and closer, and *stay* close, to α (within any desired distance ϵ).

Example:

Consider the sequence $\phi(j) = (-1)^j$, or

$$-1, 1, -1, 1, \dots .$$

This sequence does *not* converge. Intuitively, the assertion is clear; because the numbers in the sequence do not get close and stay close to any fixed value α. To verify this claim rigorously, we suppose (seeking a contradiction) that in fact the sequence *does* converge to some number α. Let $\epsilon = 1/2$. Then, by the definition of convergence, there is a positive integer K such that if $j > K$, then $|\phi(j) - \alpha| < \epsilon = 1/2$. Choose $j > K$ so that $\phi(j) = 1$, that is to say, choose j even and greater than K. Then $\phi(j + 1) = -1$.

As a result,

$$\begin{aligned} 2 &= |1-(-1)| \\ &= |\phi(j)-\phi(j+1)| \\ &= \Big|(\phi(j)-\alpha)+(\alpha-\phi(j+1))\Big| \\ &\le |\phi(j)-\alpha|+|\alpha-\phi(j+1)| \\ &< \frac{1}{2}+\frac{1}{2}=1\,. \end{aligned}$$

We have derived the untenable assertion that $2<1$. This contradiction must mean that our assumption is false: the limit number α cannot exist. So the sequence has no limit.

Example:

Consider the sequence $\phi(j) = (-1)^j/j$. Intuitively, this sequence converges. For the elements of the sequence seem to be getting smaller and smaller in absolute value, and indeed seem to tend to zero. Let us prove that this actually is the case.

Let $\epsilon > 0$. There is a natural number K so large that $1/K < \epsilon$ (this is the Archimedean property of the natural numbers). If $j > K$, then

$$|\phi(j) - 0| = |\phi(j)| = \frac{1}{j} < \frac{1}{K} < \epsilon \, ,$$

as was to be proved. So the sequence $\phi(j)$ converges to 0.

Let $\{a_j\}$ be a sequence. A *subsequence* of $\{a_j\}$ is a sequence $\{b_k\}$ whose elements come from the sequence $\{a_j\}$, in order. We usually denote the subsequence by $\{a_{j_k}\}$.

Example:

Let

$$a_1 = 1 , \ a_2 = 4 , \ a_3 = 9 , \ a_4 = 16 , \ a_5 = 25 , \ a_6 = 36 , \ \dots , \ a_j = j^2 , \ \dots$$

Then

$$a_{j_1} = 4 , \ a_{j_2} = 9 , \ a_{j_3} = 36 , \ a_{j_4} = 81 , \ \dots$$

is a subsequence. Of course a given sequence will have many different subsequences.

Basic Topological Ideas

Some of the most commonly used subsets of the real numbers are intervals. The four types of intervals are these:

open $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.
closed $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.
half-open $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.
half-open $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

A set $\mathcal{O} \subset \mathbb{R}$ is said to be *open* if, for any $x \in \mathcal{O}$, there is an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset \mathcal{O}$. A set $\mathcal{E} \subset \mathbb{R}$ is said to be *closed* if ${}^c\mathcal{E} \equiv \mathbb{R} \setminus \mathcal{E}$ is open.

A common mistake that students make is to supposed that if a set is not open then it is closed. Or if a set is not closed then it is open. This is incorrect. The set $[0, 1) = \{x \in \mathbb{R} : \mathbf{0} \leq x < 1\}$ is neither open nor closed.

Example:

Let $\mathcal{O} = \{x \in \mathbb{R} : x^2 < 1\}$. Then $\mathcal{O}$ is an open set. To see this, let $x \in \mathcal{O}$. Then certainly $|x| < 1$. Let $\epsilon = 1 - |x|$. Then we claim that $(x - \epsilon, x + \epsilon) \subset \mathcal{O}$. For if $t \in (x - \epsilon, x + \epsilon)$, then

$$|t| < |x| + |t - x| < |x| + \epsilon = |x| + (1 - |x|) = 1\,.$$

Therefore $t^2 < 1$ and $t \in \mathcal{O}$. Thus $(x - \epsilon, x + \epsilon) \subset \mathcal{O}$. As a result, $\mathcal{O}$ is open.

Example:

Let $\mathcal{E} = \{x \in \mathbb{R} : x^2 \le 1\}$. Then $\mathcal{E}$ is a closed set. To see this, we consider

$$^c\mathcal{E} = \{x \in \mathbb{R} : x < -1 \text{ or } x > 1\} .$$

Now let $x \in {}^c\mathcal{E}$. In case $x > 1$, then let $\epsilon = x - 1$. We claim that $(x - \epsilon, x + \epsilon) \subset {}^c\mathcal{E}$. For if $t \in (x - \epsilon, x + \epsilon)$, then

$$t \ge x - |x - t| > x - \epsilon = x - (x - 1) = 1 .$$

Thus $t \in {}^c\mathcal{E}$ so $(x - \epsilon, x + \epsilon) \subset {}^c\mathcal{E}$. A similar argument shows that in case $x < -1$ and $\epsilon = (-1) - x$, then $(x - \epsilon, x + \epsilon) \subset {}^c\mathcal{E}$. As a result, ${}^c\mathcal{E}$ is open; so $\mathcal{E}$ is closed.