Math 310
December 11, 2020 Lecture

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December 2, 2020

Figure: This is your instructor.

Proposition: Suppose that $\mathcal{E} \subset \mathbb{R}$ contains all its limit points. Then $\mathcal{E}$ is closed.

Proof: Suppose not. Then ${}^c\mathcal{E}$ is not open. So there is a point $y \in {}^c\mathcal{E}$ such that, for each integer $j > 0$, the interval $(y - 1/j, y + 1/j) \not\subset {}^c\mathcal{E}$. That means that, for each j, there is a point $e_j \in \mathcal{E}$ that lies in $(y - 1/j, y + 1/j)$. But then $\{e_j\} \subset \mathcal{E}$ and this sequence converges to y. Since $\mathcal{E}$ contains all its limit points, we conclude that $y \in \mathcal{E}$. That is a contradiction. $\square$

Compact Sets

A set $S \subset \mathbb{R}$ is said to be *compact* if, whenever $\{s_j\}_{j=1}^{\infty}$ is a sequence of points in S, then there is a subsequence $\{s_{j_k}\}$ that is convergent to a point of S.

Proposition:

A compact set is closed and bounded.

Proof: Let K be the compact set. If K is not bounded, then there is an element $x_1 \in K$ such that $|x_1| > 1$. Since K is unbounded, there then exists an element $x_2 \in K$ such that $|x_2| > |x_1| + 1$. Suppose, inductively, that $x_1, x_2, \ldots, x_j$ have been chosen. Then, since K is unbounded, there is an $x_{j+1} \in K$ such that $|x_{j+1}| > |x_j| + j$. It is then clear that the sequence $\{x_j\}$ contains no convergent subsequence. That contradicts the definition of compactness.

If K is not closed, then there is a sequence $\{x_j\} \subset K$ that converges to a point α that does not lie in K. But then every subsequence of $\{x_j\}$ also converges to $\alpha \notin K$. Thus K is not compact. $\square$

Proposition:

If a subset $E \subset \mathbb{R}$ is both closed and bounded, then it is compact.

Proof: Let $\{x_j\}$ be a sequence in E. Since E is bounded, it therefore lies in some interval $[-R, R]$. Let

$$S = \{x \in [-R, R] : \exists \text{ infinitely many } x_j \text{ such that } x_j \geq x\} .$$

Then S is a bounded set, for $|x| \leq R$ for every $x \in S$. Now let α be the least upper bound of S. Then α is finite, indeed $\alpha \leq R$.

We claim that there is a subsequence $\{x_{j_k}\}$ that converges to α. Let $\epsilon > 0$. By the definition of "least upper bound," there must be infinitely many of the x_j between $\alpha - \epsilon$ and $\alpha + \epsilon$, otherwise we chose the least upper bound incorrectly. This assertion is true for every $\epsilon > 0$. Thus

- We may choose x_{j_1} in $(\alpha - 1, \alpha + 1)$;
- We may choose x_{j_2} so that $j_2 > j_1$ and $x_{j_2} \in (\alpha - 1/2, \alpha + 1/2)$;
- We may choose x_{j_3} so that $j_3 > j_2$ and $x_{j_3} \in (\alpha - 1/3, \alpha + 1/3)$;

and so forth. By design, the subsequence $\{x_{j_k}\}$ converges to α. Since E is closed, we may conclude that $\alpha \in E$. Thus E is compact. □

The last two propositions taken together are known as the Heine–Borel theorem: A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

We conclude with an important result about the intersection of a nested sequence of sets. We will make decisive use of it in the next lecture. First, we give an example.

Example:

For $j = 1, 2, 3, \ldots$, let

$$U_j = \left\{ x \in \mathbb{R} : 0 < x < \frac{1}{j} \right\} .$$

Then $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$. And each U_j is uncountable. One's intuition might suggest that $\mathcal{U} \equiv \cap_j U_j$ will certainly have points in it. But that is not the case. In fact, if $x > 0$, then there is a positive integer j so large that $1/j < x$. But then $x \notin U_j$, so certainly $x \notin \mathcal{U}$. If $x \leq 0$, then x does not lie in any U_j so $x \notin \mathcal{U}$. Thus $\mathcal{U}$ is empty.

The situation is more favorable for nested sequences of compact sets:

Theorem

Let $E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$ be compact, nonempty sets. Then $\mathcal{E} \equiv \cap_j E_j \neq \emptyset$.

Proof: Let $e_1 \in E_1$, $e_2 \in E_2$, and so forth. The sequence $\{e_j\}$ lies in E_1, which is compact. So there is a subsequence $\{e_{j_k}\}$ that converges to a point $e \in E_1$. But all of the terms of this subsequence for $k \geq 2$ also lie in E_2. Since E_2 is closed, we may conclude that the limit point e is also in E_2. Continuing, we find that $e \in E_j$ for every j. As a result, $e \in \cap_j E_j \equiv \mathcal{E}$. So $\mathcal{E} \neq \emptyset$. □

The Cantor Set

We now use the ideas developed in the first three sections to demonstrate the existence of a remarkable set of real numbers that is known as the "Cantor ternary set." Note, as usual, that a careful and rigorous understanding of the real numbers is necessary in order to effect this construction.

Let $I = [0, 1]$, the unit interval in the real line. Define a sequence of nested compact sets as follows:

- $I_0 = [0, 1]$;
- $I_1 = [0, 1/3] \cup [2/3, 1]$;
- $I_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$;
- $\ldots$


I 0


I_1

I_2

The construction continues. Each set I_{j+1} is constructed by removing the middle open third from each closed interval in the set I_j. Refer to the figures for the sets I_j.

Obviously $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ and every I_j is nonempty. Also each I_j is closed and bounded, hence compact. Therefore $C = \cap_j I_j$ is a nonempty compact set. We call it *the Cantor ternary set*.

Now we prove a sequence of propositions to establish some of the fundamental properties of the Cantor ternary set.

Proposition:

Let $S = [0, 1] \setminus C$. Then S is a union of intervals with total length 1.

Of course the unit interval $[0, 1]$ itself has length 1. The proposition establishes that the complement of the Cantor set in the unit interval has length 1. These assertions together suggest that the Cantor set has length 0, so that it is "small" in some sense. In the subject of measure theory, one makes these assertions precise.

Proof of the Proposition: Of course

$$\begin{aligned}[0,1]\setminus C &= [0,1]\cap {}^cC \\ &= [0,1]\cap {}^c\Big(\bigcap_j I_j\Big) \\ &= [0,1]\bigcap\Big(\bigcup_j {}^cI_j\Big) \\ &= \bigcup_j\big([0,1]\cap {}^cI_j\big)\,.\end{aligned}$$

And each of the sets $[0,1]\cap {}^cI_j$ is a union of intervals. Thus we see explicitly that the complement of the Cantor set is a union of intervals. More is true: these complements form an increasing union. So it is easy to keep track of all the intervals and to sum up their lengths:

- First, there is a single interval of length $1/3$.
- Second, there are two intervals of length $1/9$.
- Third, there are four intervals of length $1/27$.
- ...

Thus we may add up the lengths of all the intervals in the complement of C:

$$\text{length}({}^c C) = \sum_{j=1}^{\infty} \frac{2^{j-1}}{3^j} = \frac{1}{3} \cdot \sum_{j=0}^{\infty} \left(\frac{2}{3}\right)^j = \frac{1}{3} \cdot \frac{1}{1 - 2/3} = 1 \, . \quad \square$$

So, in the sense of length, the Cantor set is small. In the next lecture we shall show that, in the sense of cardinality, the Cantor set is large.