Math 310
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Figure: This is your instructor.

Proposition:

The interior of any set S is open.

Proof: If $p \in \overset{\circ}{S}$ and U is a neighborhood of p that lies inside S, then any point x of U is also in the interior of S. For U will be the required neighborhood of x that lies in S.

Now let $S \subset X$ be any set and let p be a point of its interior. We know that there is a neighborhood U_p of p such that U_p lies entirely in S. We then know, by the last paragraph, that $U_p \subset \overset{\circ}{S}$. Therefore

$$\overset{\circ}{S} = \bigcup_{p \in \overset{\circ}{S}} \{p\} \subset \bigcup_{p \in \overset{\circ}{S}} U_p \subset \overset{\circ}{S} \,.$$

We conclude that

$$\bigcup_{p \in \mathring{S}} U_p = \mathring{S} \, .$$

The set on the left-hand side of this last expression, being the union of open sets, is open. Therefore $\mathring{S}$ is open. □

Proposition:

The boundary of any set S is closed.

Proof: Let x be a point that is not in ∂S, the boundary of S. Then there is some neighborhood U of x that does not intersect both S and ${}^cS$. It follows that any point $t \in U$ also has such a neighborhood, namely, U itself. So U lies in the complement of ∂S. It follows, by the lemma, that the complement of ∂S is open. So ∂S is closed. $\square$

Definition:

The *closure* of a set S is defined to be the intersection of all closed sets that contain S. We denote the closure of S by $\overline{S}$.

Of course the closure $\overline{S}$ of S is closed.

Proposition:

The set $\overline{S}$ equals the union of S and ∂S.

Proof: Suppose that x is a point that is not in $S \cup \partial S$. Since x is not in ∂S, there is a neighborhood U of x that either does not intersect S or does not intersect ${}^c S$. We know that $x \notin S$, so it must be that $U \subset {}^c S$. So, in fact, we see that every point of U is not in $S \cup \partial S$. Thus the complement of $S \cup \partial S$ is open, and $S \cup \partial S$ is closed. We conclude that $\overline{S} \subset S \cup \partial S$.

If instead $x \notin \overline{S}$, then, since ${}^c\overline{S}$ is open, there is a neighborhood U of x that lies in ${}^c\overline{S}$ and hence in ${}^cS$. So certainly U is disjoint from $S \cup \partial S$ and thus, in particular, $x \notin (S \cup \partial S)$. We conclude that $S \cup \partial S \subset \overline{S}$.

The two inclusions taken together give our result. $\square$

The last proposition is consistent with our intuition. In the closed disc, we see that the closure is simply the closed disc, which is the interior plus the boundary circle. In other words, the closure of S is the interior set $\overset{\circ}{S}$ union its boundary ∂S.

A commonly used term in this subject is "accumulation point." If $(X, \mathcal{U})$ is a topological space and $A \subset X$, then we say that x is an *accumulation point* of A if every neighborhood of x contains points of A other than x itself. As an instance, if $X = \mathbb{R}$ with the Euclidean topology and A is the interval $(0, 1)$, then the points 0 and 1 are accumulation points of A. In fact the set of *all* accumulation points for A is just the closed interval $[0, 1]$. We leave it to the reader to use the ideas presented here to show that a closed set in a topological space contains all its accumulation points. We can actually say more, and we state it as a formal proposition:

Proposition:

Let $(X, \mathcal{U})$ be a topological space and $S \subset X$. Then the closure of S equals the union of S and its accumulation points.

Proof: Exercise for the reader. Clearly S lies in the closure and the accumulation points lie in the closure. So you need to concentrate on the converse direction. If s lies in the closure of S and s is not an accumulation point then what can you say about s? $\square$

If X is any set, then the topology just consisting of X itself and the empty set $\emptyset$ is the smallest topology on X. We call this the trivial topology. By contrast, the topology in which each singleton $\{x\}$ for $x \in X$ is open is the largest topology on X. We call the latter topology the *discrete topology*, and we say that the space is discrete.

Mappings

Our principal tool for comparing and contrasting topological spaces will be mappings. The mappings that carry the most information for us are the continuous mappings. A *mapping* $f : X \to Y$ is a function on a set X that takes values in a space Y rather than in the real numbers or the complex numbers. We shall formulate our notion of continuity in terms of the inverse image of a mapping. Let $f : A \to B$ be a mapping. Let $S \subset B$. Then $f^{-1}(S) \equiv \{x \in A : f(x) \in S\}$. We call this set the *inverse image of the set S under the mapping f*.

Definition:

Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be topological spaces. A function (or mapping) $f : X \to Y$ is said to be *continuous* if, whenever $V \subset Y$ is open, then $f^{-1}(V) \subset X$ is open.

Remark:

This definition requires some discussion. We are working in an abstract topological space. We do not necessarily have a notion of distance, so we cannot say "if the variable x is less than δ distant from c, then $f(x)$ is less than ϵ distant from $f(c)$." We instead rely on our most fundamental structure—the open sets—to express the idea of continuity. We will have to do some work to see that the new notion of continuity, in the appropriate context, is equivalent to the old one.

Remark:

Let us recall here the rigorous definition of continuity that we learned in calculus:

Let I be an open interval in the real line and $f : I \to \mathbb{R}$ a function. Fix a point $c \in I$. We say that f is continuous at c if, for any $\epsilon > 0$, there is a $\delta > 0$ such that, whenever $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

This definition, thought about properly, makes good intuitive sense.

Proposition:

On the real line, the classical definition of continuity is equivalent to the new definition (formulated in the language of inverse images of open sets).

Proof: Suppose that I is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$. Assume that f is continuous according to the classical definition. Let V be an open subset of $\mathbb{R}$; we must show that $f^{-1}(V)$ is open. Consider the set $f^{-1}(V)$ and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and, since V is open, there exists an $\epsilon > 0$ such that the interval $(f(x) - \epsilon, f(x) + \epsilon) \subset V$. By the classical definition, there exists a $\delta > 0$ such that if $t \in (x - \delta, x + \delta)$, then $f(t) \in (f(x) - \epsilon, f(x) + \epsilon)$. This says that the interval $(x - \delta, x + \delta)$ lies in $f^{-1}(V)$. Hence $f^{-1}(V)$ is open. We have shown that the inverse image of an open set is open, and that is the new definition of continuity.

For the converse, assume that $f : I \to \mathbb{R}$ satisfies the new definition of continuity given above. Fix a point $x \in I$. Let $\epsilon > 0$. The interval $(f(x) - \epsilon, f(x) + \epsilon)$ is an open subset of the range $\mathbb{R}$. Thus, by hypothesis, the inverse image $f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$ is open. It is an open neighborhood of the point x. So there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset f^{-1}((f(x) - \epsilon, f(x) + \epsilon))$. This means that if $|t - x| < \delta$, then $|f(t) - f(x)| < \epsilon$, which is the classical definition of continuity. $\square$

One thing that is nice about our new definition of continuity is that it is simple and natural to use in situations where the traditional definition would be awkward. We examine some examples.

Example:

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Discuss the continuity of f.

We know from experience that f is continuous—all polynomial functions are continuous. But it is instructive to examine the new definition of continuity in this context.

Let V be an open subset of the range space $\mathbb{R}$. We may take V to be an interval $I = (a, b)$ since any open set is a union of such intervals (exercise). Then

- If $0 \leq a < b$, then $f^{-1}(I) = (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b})$, and that is an open set.
- If $a < 0 \leq b$, then $f^{-1}(I) = (-\sqrt{b}, \sqrt{b})$, and that is an open set.
- If $a < b < 0$ then $f^{-1}(I) = \emptyset$, and that is an open set.

We have verified directly that f is continuous according to the new definition.

Because the function f in the last example is not uniformly continuous, it is actually quite tedious to verify the continuity of f using the old calculus definition of continuity. As you can see, with the new definition it is rather straightforward.

Example:

Let us equip the rational numbers $\mathbb{Q}$ with the topology that the number system inherits from the superset $\mathbb{R}$. This means that a set $V \subset \mathbb{Q}$ is open precisely when there is an open set $U \subset \mathbb{R}$ such that $U \cap \mathbb{Q} = V$.

Consider the function $f : \mathbb{Q} \to \mathbb{Q}$ that is defined as follows. If p/q is a rational number expressed in lowest terms (i.e., p and q have no prime factors in common), with q positive, then set $f(p/q) = 1/q$. Determine whether f is continuous.

In fact, f is discontinuous. The values of f are 1/1, 1/2, 1/3, 1/4, etc. Let us take a neighborhood V of 1/2 in the image—this is a typical open set. We take the neighborhood to be an interval that in $\mathbb{Q}$ is small enough that it does not contain any of the other image points (1/1, 1/3, 1/4, etc.). We see that

$$f^{-1}(V) = \left\{ \ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \right\}.$$

This is not an open set because the point $5/2$ is isolated—it has no interval about it that lies in the set. So f is not continuous.