Math 310 Fall, 2020
Krantz

FINAL EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

You will submit your work on the CrowdMark system. Be sure to answer each question on a separate sheet of paper. You can answer questions 3 and 4 (each with three parts) on one page. Scan in each piece of paper separately.

This exam is worth 100 points. It is 25% of your grade.

Be sure to ask questions if anything is unclear.

(6 points) **1.** Are the statements $(\sim \mathbf{A} \wedge \sim \mathbf{B}) \Rightarrow (\mathbf{A} \vee \mathbf{B})$ and $\sim (\mathbf{A} \vee \mathbf{B}) \Rightarrow (\mathbf{A} \vee \mathbf{B})$ logically equivalent?

(6 points) **2.** Draw a Venn diagram to illustrate the identity

$$S \setminus (T \cup U) = (S \setminus T) \cap (S \setminus U) .$$

(9 points) **3.** Which of these functions is one-to-one? Which is onto? Give a brief reason for each answer.

(a) $f : \mathbb{R} \to \mathbb{R}$ $\qquad f(x) = x^2 + x$

(b) $g : \mathbb{Z} \to \mathbb{Z}$ $\qquad g(n) = n(n-1)$

(c) $h : \mathbb{R} \to \mathbb{R}$ $\qquad h(x) = x \cos x$

(9 points) **4.** Which of these sets is countable and which uncountable (give a brief reason for each answer)?

- **(a)** $\mathbb{C} \setminus \mathbb{R}$
- **(b)** $\mathbb{Z} \times \mathbb{R}$
- **(c)** $\mathbb{Z} \times \mathbb{N}$

(6 points) **5.** Prove that is impossible for two successive positive integers to both be perfect squares.

(6 points) **6.** Use mathematical induction to prove that

$$1^2 + 2^2 + \cdots n^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

(8 points) **7.** Construct a Cantor-like set that has length 1/3.

(6 points) **8.** Prove that $\sqrt{6}$ is irrational.

(8 points) **9.** Prove that multiplication of rational numbers is well defined.

(8 points) **10.** What is the multiplicative inverse of the complex number i?

(8 points) **11.** Find all cube roots of the complex number $1 - i$.

(7 points) **12.** Let X be the integers $\mathbb{Z}$. Declare a set $U \subset X$ to be open if the complement of U is finite or empty or if U itself is empty. Show that this defines a topology on X.

(7 points) **13.** Let $E \subset \mathbb{R}$ be a closed set and $K \subset \mathbb{R}$ a compact set. Assume that $E \cap K = \emptyset$. Prove that there is an $\epsilon > 0$ so that if $e \in E$ and $k \in K$ then $|e - k| > \epsilon$.

(6 points) **14.** What is the multiplicative inverse of the quaternion $\mathbf{i} + \mathbf{j}$?