MATH 310



Solutions to Practice Final Exam

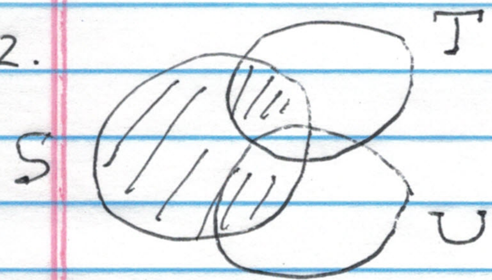
1.

| $A$ | $B$ | $C$ | $\sim A$ | $\sim B$ | $\sim A \vee \sim B$ | $B \wedge C$ | $A \wedge B$ | $(\sim A \vee \sim B) \vee (B \wedge C)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | F | F | T | T | T |
| T | F | T | F | T | T | F | F | T |
| F | T | T | T | F | T | T | F | T |
| F | F | T | T | T | T | F | F | T |
| T | T | F | F | F | F | F | T | F |
| T | F | F | F | T | T | F | F | T |
| F | T | F | T | F | T | F | F | T |
| F | F | F | T | T | T | F | F | T |

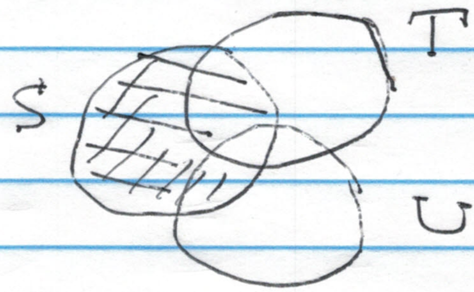
| $(A \wedge B) \Rightarrow (B \wedge C)$ |
|---|
| T |
| T |
| T |
| T |
| F |
| T |
| T |
| T |

These are the same, so logically equivalent.

2.



$S \setminus (T \cap U)$



$(S \setminus T) \cup (S \setminus U)$

3. a) $f(0) = f(2)$ so not one-to-one.
The graph is an upward opening parabola, so not onto.

b) $g(0) = g(-2)$, so not one-to-one.
The quadratic formula shows that there is no $n$ such that $g(n) = -5$, so not onto.

c) $h(0) = h(\pi)$, so not one to one.
The function is continuous and takes arbitrarily large positive values and arbitrarily large negative values. So the intermediate value property tells us that $h$ is onto.

4. (a) Let $X = \{iy : y \in \mathbb{R}\}$. Then $X$ is uncountable.
And

$$\Phi : X \to \mathbb{C} \setminus \mathbb{Z}$$
$$iy \to 0 + iy$$

is one-to-one. So $\operatorname{card}(X) \leq \operatorname{card}(\mathbb{C} \setminus \mathbb{Z})$.
We conclude that $\mathbb{C} \setminus \mathbb{Z}$ is uncountable.

(b) The function

$$\Phi : \mathbb{C} \to \mathbb{Z} \times \mathbb{C}$$
$$z \to (0, z)$$

is one-to-one. So $\mathrm{card}(\mathbb{C}) \leq \mathrm{card}(\mathbb{Z} \times \mathbb{C})$. Of course $\mathbb{C}$ is uncountable. Hence $\mathbb{Z} \times \mathbb{C}$ is uncountable.

(c) The product of two countable sets is countable. So $\mathbb{Q} \times \mathbb{N}$ is countable.

5. If $n^3$ is a perfect cube then $(n+1)^3$ is the next perfect cube. But

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

and

$$(n+1)^3 - n^3 = (n^3 + 3n^2 + 3n + 1) - n^3 = 3n^2 + 3n + 1$$
$$> 6.$$

So successive cubes differ by ~~at~~ least 6, not by 3.

6. $P(n)$ is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

$P(1)$ is $1 = \frac{1 \cdot (1+1)}{2} = 1$ which is true.

Now assume $P(n)$. So

$$1+\cdots+n = \frac{n(n+1)}{2}$$

Add $(n+1)$ to both sides:

$$1+\cdots+n+(n+1) = \frac{n(n+1)}{2}+n+1$$

$$= \frac{n^2+n+2n+2}{2}$$

$$= \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}.$$

This is $P(n+1)$ so the induction is complete.

7. Construct a Cantor-like set by removing one interval of length $\frac{1}{4}$, two intervals of length $\left(\frac{1}{4}\right)^2$, four intervals of length $\left(\frac{1}{4}\right)^3$, and so forth. So the total length of all removed intervals is

$$\sum_{j=1}^{\infty} 2^{j-1}\cdot\left(\frac{1}{4}\right)^j = \frac{1}{4}\sum_{j=1}^{\infty} 2^{j-1}\left(\frac{1}{4}\right)^{j-1} = \frac{1}{4}\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{j-1}$$

$$= \frac{1}{4}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^j = \frac{1}{4}\,\frac{1}{1-1/2} = \frac{1}{4-2} = \frac{1}{2}.$$

Thus the length of the remaining Cantor set is $1/2$.

8. Assume to the contrary that 8 has a rational square root $a/b$ in lowest terms.

Then
$$\left(\frac{a}{b}\right)^2 = 8$$
$$a^2 = 8b^2. \quad (*)$$

The right hand side is divisible by three factors of 2 hence the left hand side is divisible by three factors of 2. So two factors of 2 must divide $a$. In other words, $4 \mid a$ or $a = 4\alpha$, some $\alpha$. Then $(*)$ says that

$$(4\alpha)^2 = 8b^2$$
$$16\alpha^2 = 8b^2$$
$$2\alpha^2 = b^2$$

Since 2 divides the left hand side then 2 divides the right hand side so 2 divides $b$. But now we have shown that 2 divides $a$ and 2 divides $b$. That is a contradiction. So the rational square root cannot exist.

9. Let $S = \{s \in \mathbb{R} : s > 0 \text{ and } s^2 < 2\}$.

Then $S \neq \emptyset$ since $1 \in S$. Also

$S$ is bounded above by 2.

Thus $S$ has a least upper bound $y$. Obviously $y \geq 1$

We claim $y^2 = 2$.

If in fact $y^2 < 2$, then set

$\varepsilon = \frac{2-y^2}{12}$. Then $\varepsilon > 0$ and

$$
\begin{aligned}
(y+\varepsilon)^2 &= y^2 + 2y\varepsilon + \varepsilon^2 \\
&= y^2 + 2 \cdot y \cdot \frac{2-y^2}{12} + \frac{2-y^2}{12} \cdot \frac{2-y^2}{12} \\
&< y^2 + \frac{y \cdot (2-y^2)}{6} + \frac{2 \cdot (2-y^2)}{144} \\
&< y^2 + \frac{2-y^2}{2} + \frac{2-y^2}{72} \\
&< 2
\end{aligned}
$$

Thus $y + \varepsilon \in S$ and $y$ cannot be an upper bound for $S$.

A similar argument shows that $y^2$ cannot be greater than 2. So $y^2 = 2$ and 2 has a square root.

10. It is $\dfrac{\overline{-i}}{|-i|^2} = \dfrac{i}{1} = i$.

11. $i = 1\cdot e^{i\pi/2}$. So

$$r^3 e^{3i\theta} = 1\cdot e^{i\pi/2}$$

$$r = 1,\ \theta = \frac{\pi}{6}, \quad z_1 = 1\cdot e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$r^3 e^{3i\theta} = 1\cdot e^{i5\pi/2}$$

$$r = 1,\ \theta = \frac{5\pi}{6}, \quad z_2 = 1\cdot e^{i5\pi/6} = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$r^3 e^{3i\theta} = 1\cdot e^{i9\pi/2}$$

$$r = 1,\ \theta = 3\pi/2, \quad z_3 = 1\cdot e^{i3\pi/2} = -i.$$

12. This is not a topology. For

$E$ = even integers is open

$O$ = odd integers is open

But $E \cup O = \mathbb{Z}$ does not have infinite complement so not open.

13. Let
$$f(x) = \begin{cases} (x-1)^4 & \text{for } x > 1 \\ 0 & \text{for } -1 \le x \le 1 \\ (x+1)^4 & \text{for } x < -1. \end{cases}$$

Then
$$f'(x) = \begin{cases} 4(x-1)^3 & \text{for } x > 1 \\ 0 & \text{for } -1 \le x \le 1 \\ 4(x+1)^3 & \text{for } x < -1 \end{cases}$$

So $f'$ is continuous. Also

$$f''(x) = \begin{cases} 12(x-1)^2 & \text{for } x > 1 \\ 0 & \text{for } -1 \le x \le 1 \\ 12(x+1)^2 & \text{for } x < -1 \end{cases}$$

So $f''$ is continuous.

14. $(\underline{\underline{j}} - \underline{\underline{k}}) \cdot (a \cdot \underline{\underline{1}} + b\underline{\underline{i}} + c\underline{\underline{j}} + d\underline{\underline{k}}) = \underline{\underline{1}}$

$a\underline{\underline{j}} - b\underline{\underline{k}} - c\underline{\underline{1}} + d\underline{\underline{i}} - a\underline{\underline{k}} - b\underline{\underline{j}} + c\underline{\underline{i}} + d\underline{\underline{1}} = \underline{\underline{1}}$

$(-c+d)\underline{\underline{1}} + (d+c)\underline{\underline{i}} + (a-b)\underline{\underline{j}} + (-b-a)\underline{\underline{k}} = \underline{\underline{1}}$.

So
$$\left.\begin{array}{l} -c+d = 1 \\ d+c = 0 \end{array}\right\} \; c = -\tfrac{1}{2},\ d = \tfrac{1}{2}$$
$$\left.\begin{array}{l} a-b = 0 \\ -b-a = 0 \end{array}\right\} \; a = b = 0$$

So the multiplicative inverse of $\underline{\underline{j}} - \underline{\underline{k}}$ is $-\frac{1}{2}\underline{\underline{j}} + \frac{1}{2}\underline{\underline{k}}$.