SOLUTIONS TO HW 2

pp. 60 - 65

1. Let $j = 2m+1$,
$k = 2n+1$.

Then
$$j \cdot k = (2m+1)\cdot(2n+1) = 4mn + 2m + 2n + 1$$
$$= 2(2mn + m + n) + 1,$$
which is odd.

3. $P(n)$ is $1^2 + 2^2 + \cdots + n^2 = \dfrac{2n^3 + 3n^2 + n}{6}$.

$P(1)$ is $1^2 = \dfrac{2\cdot 1^3 + 3\cdot 1^2 + 1}{6}$, which is true.

Now assume $P(n)$. So

$$1^2 + 2^2 + \cdots + n^2 = \frac{2n^3 + 3n^2 + n}{6}$$

Add $(n+1)^2$ to both sides:

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{2n^3 + 3n^2 + n}{6} + (n+1)^2$$
$$= \frac{2n^3 + 3n^2 + n + 6n^2 + 12n + 6}{6}$$
$$= \frac{2n^3 + 9n^2 + 13n + 6}{6}$$
$$= \frac{2(n+1)^3 + 3(n+1)^2 + (n+1)}{6}.$$

That is $P(n+1)$.

12. Assume that $n = m^2$. The next perfect square is $(m+1)^2 = m^2 + 2m + 1$

$$> n + 1.$$

So $n+1$ is not a perfect square.

13. Assume that $j \cdot k$ is even but neither j nor k is even. Then j, k are both odd.

So $j = 2m+1$

$k = 2n+1$

$$j \cdot k = (2m+1)(2n+1) = 4mn + 2m + 2n + 1$$
$$= 2(2mn + m + n) + 1$$

which is odd. Contradiction.

17. There are infinitely many Pythagorean triples:

$3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, ... etc.

But $2^2 + 3^2$ is <u>not</u> a perfect square.

20. $m^2 - n^2 = (m-n)(m+n)$. This will <u>not</u> be a prime unless $m = 2$, $n = 1$. Then

$m^2 - n^2 = 3$. Same when $m = n+1$.

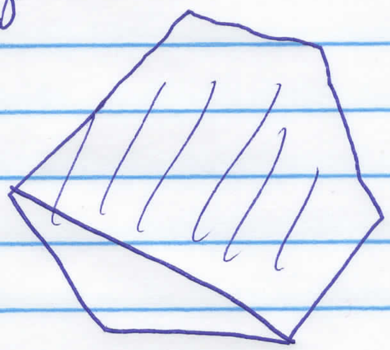
24. Let $\alpha < \beta$ be distinct real numbers. So $\beta - \alpha > 0$. Choose a natural number N such that

$$\frac{1}{N} < \beta - \alpha.$$

Look at the numbers $\frac{j}{N}$ for $j \in \mathbb{Z}$.

One of these numbers must lie between α and β.

28. $k=3$ is the case of a triangle. We know that the sum of the angles in a triangle is $180°$. Now let W be a convex polygon with $k+1$ sides. Cut off a triangle as in the picture:



The shaded region has k sides, so the sum of its angles is $(k-2)\cdot 180$. But the triangle has angles adding up to $180°$. So the total of all the angles

is $$(k-2)\cdot 180 + 180 = (k-1)\cdot 180$$
$$= ((k+1)-2)\cdot 180.$$

34. $P(n)$ is $2^n > n^2+1$

$P(5)$ is $2^5 > 25+1$ or $32 > 26$, which is true.

Now assume $$2^n > n^2+1$$

Then $$2^{n+1} > 2n^2+2$$
$$= n^2+n^2+2$$
$$> n^2+2n+2 = (n+1)^2+1.$$

This is $P(n+1)$.

35. $P(n)$ is
If you put $(n+2)$ letters in n mailboxes then some box will contain two letters.
$P(1)$ say that if you put 2 letters in one box then some box will contain two letters.
Now suppose that $P(n)$ is true. Let us examine $P(n+1)$. So we have $(n+2)$ letters and $n+1$ mailboxes. Look at the first n boxes. If $(n+2)$ letters go in the first n boxes, then the inductive hypothesis tells us that some box receives two letters.
If only n letters go into the first n boxes then the last box receives 2 letters. So we are done.

37. If letter a goes into envelope B. Then envelope A must also receive a wrong letter. So it is impossible for there to be just one wrong letter in an envelope.

42. $P(n)$ is $n^3 - n$ is divisible by 6.
$n=2$: $2^3 - 2 = 6$ is divisible by 6.

Assume $P(n)$. So

$n^3 - n$ is divisible by 6.

Now examine $P(n+1)$. So

$$(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - (n+1)$$
$$= n^3 + 3n^2 + 2n$$
$$= (n^3 - n) + 3n^2 + 3n$$
$$= (n^3 - n) + 3n(n+1) \qquad (*)$$

By the inductive hypothesis, $n^3 - n$ is divisible by 6. And either n or $n+1$ is even. So $3n(n+1)$ is divisible by 6.

Hence $(*)$ is divisible by 6.

44. We will prove that

$$P(n): \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

$n = 1$ is obvious.

Assume $P(n)$:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$$

Add $\frac{1}{\sqrt{n+1}}$ to both sides. So

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n} + \frac{1}{\sqrt{n+1}}.$$

We need to see that

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} \geq \sqrt{n+1}.$$

Squaring both sides, this is

$$n + \frac{1}{n+1} + 2\frac{\sqrt{n}}{\sqrt{n+1}} \geq n+1$$

$$\frac{1}{n+1} + \frac{2\sqrt{n}}{\sqrt{n+1}} \geq 1$$

$$1 + 2\sqrt{n}\sqrt{n+1} \geq n+1$$

$$2\sqrt{n(n+1)} \geq n$$

$$4n(n+1) \geq n^2.$$

This is true.