MATH 310 SOLUTIONS TO HW6

Ch. 6

11. It is impossible to divide 1 by 5, for instance.
Here
$$1 = (1,2)$$
$$5 = (1,6).$$
If there did exist an $x = (a,b)$ such that
$$5 \cdot x = 1$$
then
$$(1,6)\cdot(a,b) = (1,2)$$
so
$$(b+6a,\ a+6b) = (1,2)$$
$$\begin{aligned} 6a+b &= 1 \\ a+6b &= 2 \end{aligned} \quad \begin{aligned} &\Rightarrow \quad 6a+b = 1 \\ &\Rightarrow \quad 6a+36b = 12 \end{aligned}$$
$$35b = 11$$
$$b = \frac{11}{35}$$
$$6a + \frac{11}{35} = 1$$
$$6a = \frac{24}{35}$$
$$a = \frac{4}{35}$$
So a, b are not natural numbers. That is a contradiction.

15. If $\sqrt{2}+\sqrt{3}$ were rational then

$$\sqrt{2}+\sqrt{3} = \frac{p}{q} \text{ in lowest terms.}$$

Squaring,

$$2 + 2\sqrt{6} + 3 = \frac{p^2}{q^2}$$

$$2\sqrt{6} = \frac{p^2}{q^2} - 5$$

$$\sqrt{6} = \frac{p^2}{2q^2} - \frac{5}{2}$$

so $\sqrt{6}$ is rational. But that is false.

17. The complex numbers are not defined as equivalence classes.

23. $1 = 1 \cdot e^{i0}$

$(re^{i\theta})^3 = 1 \cdot e^{i0}$

$r^3 e^{3i\theta} = 1 \cdot e^{i0} \Rightarrow r = 1,\ \theta = 0 \Rightarrow z = 1 \cdot e^{i0} = 1$

is a cube root.

$(re^{i\theta})^3 = 1 \cdot e^{i(0+2\pi)} \Rightarrow r = 1,\ \theta = \frac{2\pi}{3} \Rightarrow z = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

is a cube root.

$(re^{i\theta})^3 = 1 \cdot e^{i(0+4\pi)} \Rightarrow r = 1,\ \theta = \frac{4\pi}{3} \Rightarrow z = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

is a cube root.

27. $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$$re^{i(\theta+2\pi)} = r(\cos(\theta+2\pi) + i\sin(\theta+2\pi))$$

$$= r\cos\theta + i\sin\theta = z$$

by the periodicity of sine and cosine.

29. $1 + i = \sqrt{2}\, e^{i\pi/4}$.

$r^4 e^{i4\theta} = \sqrt{2}\, e^{i\pi/4} \Rightarrow r = 2^{1/8},\ \theta = \frac{\pi}{16},\ z = 2^{1/8} e^{i\pi/16}$

$r^4 e^{i4\theta} = \sqrt{2}\, e^{i9\pi/4} \Rightarrow r = 2^{1/8},\ \theta = \frac{9\pi}{16},\ z = 2^{1/8} e^{i9\pi/16}$

$r^4 e^{i4\theta} = \sqrt{2}\, e^{i17\pi/4} \Rightarrow r = 2^{1/8},\ \theta = \frac{17\pi}{16},\ z = 2^{1/8} e^{i17\pi/16}$

$r^4 e^{i4\theta} = \sqrt{2}\, e^{i25\pi/4} \Rightarrow r = 2^{1/8},\ \theta = \frac{25\pi}{16},\ z = 2^{1/8} e^{i25\pi/16}$

40. $z = x + iy,\ w = u + iv,\ c = a + ib$

$c\cdot(z + w) = (a+ib)\cdot((x+u) + i(y+v))$

$= a(x+u) - b(y+v) + i(a(y+v) + b(x+u))$

$= [(ax - by) + i(ay + bx)] + [(au - bv) + i(av + bu)]$

$= c\cdot z + c\cdot w$

43. $z\cdot\bar{z} = (z_1\mathbf{1} + z_2\mathbf{i} + z_3\mathbf{j} + z_4\mathbf{k})\cdot(z_1\mathbf{1} - z_2\mathbf{i} - z_3\mathbf{j} - z_4\mathbf{k})$

$$\begin{aligned}
&= z_1^2\,\mathbf{1}\cdot\mathbf{1} - z_1z_2\,\mathbf{1}\cdot\mathbf{i} - z_1z_3\,\mathbf{1}\cdot\mathbf{j} - z_1z_4\,\mathbf{1}\cdot\mathbf{k}\\
&\quad + z_2z_1\,\mathbf{i}\cdot\mathbf{1} - z_2^2\,\mathbf{i}\cdot\mathbf{i} - z_2z_3\,\mathbf{i}\cdot\mathbf{j} - z_2z_4\,\mathbf{i}\cdot\mathbf{k}\\
&\quad + z_3z_1\,\mathbf{j}\cdot\mathbf{1} - z_3z_2\,\mathbf{j}\cdot\mathbf{i} - z_3^2\,\mathbf{j}\cdot\mathbf{j} - z_3z_4\,\mathbf{j}\cdot\mathbf{k}\\
&\quad + z_4z_1\,\mathbf{k}\cdot\mathbf{1} - z_4z_2\,\mathbf{k}\cdot\mathbf{i} - z_4z_3\,\mathbf{k}\cdot\mathbf{j} - z_4^2\,\mathbf{k}\cdot\mathbf{k}\\
&= z_1^2\cdot\mathbf{1} - \cancel{z_1z_2}\,\mathbf{i} - \cancel{z_1z_3}\,\mathbf{j} - \cancel{z_1z_4}\,\mathbf{k}\\
&\quad + \cancel{z_2z_1}\,\mathbf{i} + z_2^2\,\mathbf{1} - \cancel{z_2z_3}\,\mathbf{k} + \cancel{z_2z_4}\,\mathbf{j}\\
&\quad + \cancel{z_1z_3}\,\mathbf{j} + \cancel{z_2z_3}\,\mathbf{k} + z_3^2\,\mathbf{1} - \cancel{z_3z_4}\,\mathbf{i}\\
&\quad + \cancel{z_1z_4}\,\mathbf{k} - \cancel{z_2z_4}\,\mathbf{j} + \cancel{z_4z_3}\,\mathbf{i} + z_4^2\,\mathbf{1}\\
&= (z_1^2 + z_2^2 + z_3^2 + z_4^2)\,\mathbf{1}.
\end{aligned}$$

45. By the Euclidean algorithm,

$$p(z) = (z-\alpha)\cdot q(z) + r(z)$$

with degree r < degree $(z-\alpha)$. So degree $r = 0$. Hence r is a constant. But then

$$0 = p(\alpha) = (\alpha-\alpha)\cdot q(\alpha) + r(\alpha)$$

so $r = 0$. Hence $p(z) = (z-\alpha)q(z)$ and $(z-\alpha)$ divides p.