SOLUTIONS

Math 310
Krantz

SECOND MIDTERM EXAM

General Instructions: Read the statement of each problem carefully. If you want full credit on a problem then you must show your work. If you only write the answer then you will *not* receive full credit.

Be sure to ask questions if anything is unclear. This exam has 8 questions and is worth 100 points. You will have 50 minutes to take this exam.

(8 points) 1. Let $S = \{2, 3, 4, 6\}$ and $T = \{1, 3, 5, 7\}$. What is $S \cup T$? What is $S \cap T$? What is $S \setminus T$?

$S \cup T = \{1, 2, 3, 4, 5, 6, 7\}$

$S \cap T = \{3\}$

$S \setminus T = \{2, 4, 6\}$

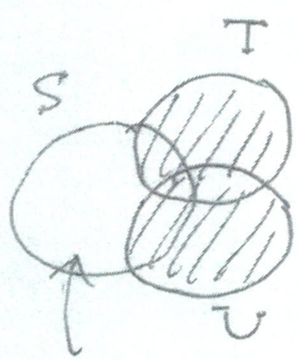
(8 points) 2. Let $S = \{2, 4, 6\}$ and $T = \{a, b, c, d\}$. What is $S \times T$? What is $T \times S$?

$S \times T = \{(2,a), (2,b), (2,c), (2,d), (4,a), (4,b), (4,c), (4,d),$
$(6,a), (6,b), (6,c), (6,d)\}$

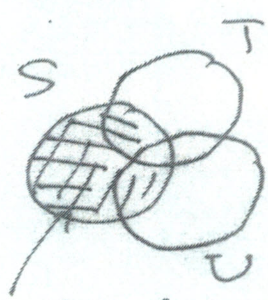
$T \times S = \{(a,2), (a,4), (a,6), (b,2), (b,4), (b,6), (c,2), (c,4),$
$(c,6), (d,2), (d,4), (d,6)\}$

(8 points) 3. Draw a Venn diagram to illustrate the identity

$$S \setminus (T \cup U) = (S \setminus T) \cap (S \setminus U).$$



$S \setminus (T \cup U)$



$(S \setminus T) \cap (S \setminus U)$.

(6 points) 4. What is the power set of $\{3, \alpha, x\}$?

$\{\{3\}, \{\alpha\}, \{x\}, \{3, \alpha\}, \{3, x\}, \{\alpha, x\}, \{3, \alpha, x\}, \emptyset\}$

(12 points) 5. Which of these functions is one-to-one? Which is onto (give a brief reason for each answer)?

(a) $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^3 + x$

$f'(x) = 3x^2 + 1 > 0$ f is increasing so one-to-one.

$\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to +\infty} f(x) = +\infty$. So f is onto.

(b) $g : \mathbb{N} \to \mathbb{N}$ $\quad g(n) = n(n + 1)$

g is increasing so one-to-one.

There is no n such that $g(n) = 1$, So not onto.

(c) $h : \mathbb{R} \to \mathbb{R}$ $\quad h(x) = x \sin x$

$h(0) = h(\pi) = 0$ so not one-to-one

h is continuous and takes arbitrarily large positive and arbitrarily large negative values. So onto.

(12 points) **6.** Which of these sets is countable and which uncountable (give a brief reason for each answer)?

(a) $\mathbb{C} \times \mathbb{R}$

$\varphi : \mathbb{R} \to \mathbb{C} \times \mathbb{R}$
$x \to (0, x)$.

So $\text{card}(\mathbb{R}) \leq \text{card}(\mathbb{C} \times \mathbb{R})$
hence uncountable

(b) $\mathbb{Z} \times \mathbb{N}$ The product of two countable sets is countable. So countable

(c) $\mathbb{Z} \times \mathbb{C}$

$h : \mathbb{C} \to \mathbb{Z} \times \mathbb{C}$
$z \mapsto (0, z)$

So $\text{card}(\mathbb{C}) \leq \text{card}(\mathbb{Z} \times \mathbb{C})$
hence uncountable

(10 points) **7.** Calculate the inverse of the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if} \quad x \leq 0 \\ x^2 & \text{if} \quad x > 0\,. \end{cases}$$

$$f^{-1}(x) = \begin{cases} x & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

(6 points) **8.** Prove that the collection of S of irrational numbers is uncountable.

If the set of irrational numbers were countable then

$$\mathbb{R} = \mathbb{Q} \cup \{\text{irrationals}\}$$

would be countable. Contradiction.

So the set of irrationals is uncountable.

(8 points) **9.** Explain why the product of a countable set and an uncountable set is uncountable.

Let S be countable and T uncountable.

Let $s_0 \in S$. Define $f : T \to S \times T$

$$t \to (s_0, t)$$

Then f is one-to-one. So

$$\operatorname{card}(T) \le \operatorname{card}(S \times T).$$

Since T is uncountable, we conclude that $S \times T$ is uncountable.

(6 points) **10.** Explain why the union of a countable set and an uncountable set is uncountable.

Let S be countable and T be uncountable.

Then $T \subset S \cup T$. So

$$\operatorname{card}(T) \le \operatorname{card}(S \cup T).$$

Since T is uncountable, we conclude that $S \cup T$ is uncountable.

(8 points) **11.** Prove that addition in the integers is well defined. You should use the actual, *rigorous* definition of the integers (in terms of ordered pairs of natural numbers) to do this problem.

Let $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$. We need to see that $(a+c, b+d) \sim (a'+c', b'+d')$. We know that $a+b' = a'+b$ and $c+d' = c'+d$. Adding left sides and right sides gives

$$a+b'+c+d' = a'+b+c'+d$$

or $(a+c)+(b'+d') = (a'+c')+(b+d)$.

So $(a+c, b+d) \sim (a'+c', b'+d')$ as desired.

(8 points) **12.** What is the multiplicative inverse of the complex number $2 - 3i$?

$$\frac{2+3i}{2^2+3^2} = \frac{2}{13} + \frac{3}{13}i$$

(8 points) **13.** Find a square root in the quaternions of the quaternion $4 \cdot \mathbf{1} + 4 \cdot \mathbf{k}$.

Guess a square root of the form $\alpha \cdot \underline{1} + \beta \cdot \underline{k}$.

So $(\alpha \cdot \underline{1} + \beta \cdot \underline{k}) \cdot (\alpha \cdot \underline{1} + \beta \cdot \underline{k}) = 4 \cdot \underline{1} + 4 \cdot \underline{k}$.

$(\alpha^2 - \beta^2) \cdot \underline{1} + 2\alpha\beta \cdot \underline{k} = 4 \cdot \underline{1} + 4 \cdot \underline{k}$

$\alpha^2 - \beta^2 = 4, \ 2\alpha\beta = 4 \Rightarrow \alpha = \frac{2}{\beta}$.

$\left(\frac{2}{\beta}\right)^2 - \beta^2 = 4 \Rightarrow 4 - \beta^4 = 4\beta^2 \Rightarrow \beta^4 + 4\beta^2 - 4 = 0$

$\beta^2 = \frac{-4 \pm \sqrt{16+16}}{2} = \frac{-4 \pm 4\sqrt{2}}{2} = -2 \pm 2\sqrt{2}$.

$\beta = \sqrt{-2+2\sqrt{2}}, \ \alpha = \frac{2}{\sqrt{-2+2\sqrt{2}}}$.

Solution $\frac{2}{\sqrt{-2+2\sqrt{2}}} \cdot \underline{1} + \sqrt{-2+2\sqrt{2}} \cdot \underline{k}$

(6 points) **14.** Find all cube roots of the complex number i.

$i = e^{i\frac{\pi}{2}}$

Solve $r^3 e^{3i\theta} = 1 \cdot e^{i\frac{\pi}{2}} \Rightarrow r = 1, \theta = \frac{\pi}{6}$.

$z_1 = 1 \cdot e^{i\frac{\pi}{6}}$

Solve $r^3 e^{3i\theta} = 1 \cdot e^{i\frac{5\pi}{2}} \Rightarrow r = 1, \theta = \frac{5\pi}{6}$

$z_2 = 1 \cdot e^{i\frac{5\pi}{6}}$

Solve $r^3 e^{3i\theta} = 1 \cdot e^{i\frac{9\pi}{2}} \Rightarrow r = 1, \theta = \frac{9\pi}{6} = \frac{3\pi}{2}$

$z_3 = 1 \cdot e^{i\frac{3\pi}{2}}$