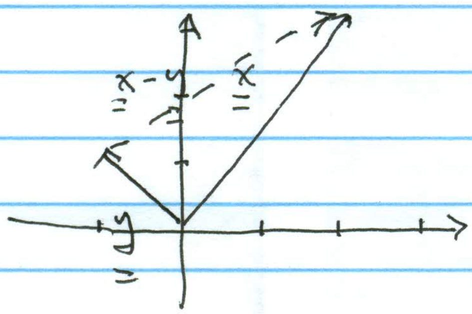
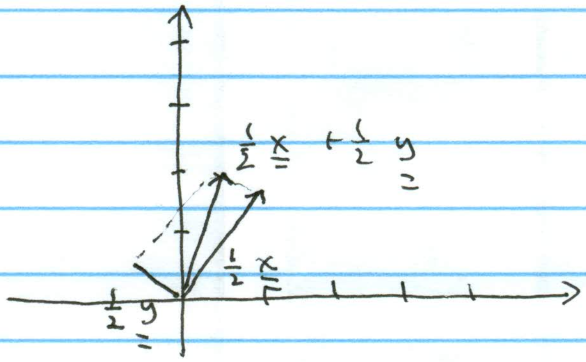
Sols. to HW 1

§1.1

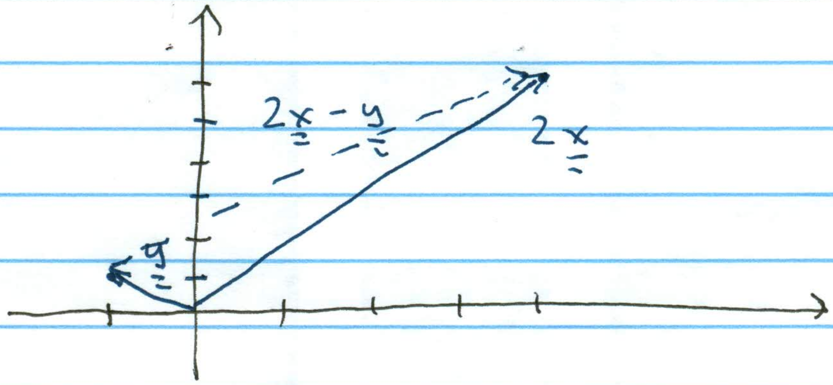
1. b) $\underline{\underline{x}} - \underline{\underline{y}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



d) $\frac{1}{2}\underline{\underline{x}} + \frac{1}{2}\underline{\underline{y}} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2 \end{bmatrix}$



f) $2\underline{\underline{x}} - \underline{\underline{y}} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$



2. Let $\underline{\underline{x}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\underline{\underline{y}} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$, $\underline{\underline{z}} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$.

One edge could be $\underline{\underline{y}} - \underline{\underline{x}} = \langle 1, 2, 2 \rangle$

Then let $\underline{\underline{w}} = \underline{\underline{z}} + \langle 1, 2, 2 \rangle = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$.

Then $\underline{\underline{x}}, \underline{\underline{y}}, \underline{\underline{z}}, \underline{\underline{w}}$ are vertices of a parallelogram.

Or one edge could be $\underline{\underline{z}} - \underline{\underline{x}} = \langle 2, -1, 4 \rangle$.

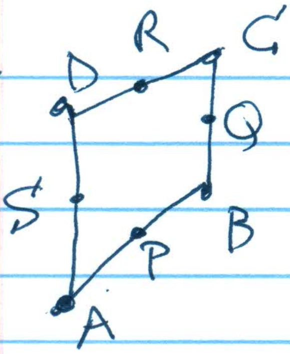
Then let $\underline{\underline{w}} = \underline{\underline{y}} + \langle 2, -1, 4 \rangle = \langle 4, 3, 7 \rangle$

Then $\underline{\underline{x}}, \underline{\underline{y}}, \underline{\underline{z}}, \underline{\underline{w}}$ are the vertices of a parallelogram.

Or one edge could be $\underline{\underline{z}} - \underline{\underline{y}} = \langle 1, -3, 2 \rangle$.

Then let $\underline{\underline{w}} = \underline{\underline{x}} + \langle 1, -3, 2 \rangle = \langle 2, -1, 3 \rangle$.

Then $\underline{\underline{x}}, \underline{\underline{y}}, \underline{\underline{z}}, \underline{\underline{w}}$ are the vertices of a parallelogram.

5. 

By Exercise 4,

$$\vec{SR} = \tfrac{1}{2}\vec{AC}$$

$$\vec{PQ} = \tfrac{1}{2}\vec{AC}$$

So $\vec{SR}$ and $\vec{PQ}$ are parallel.

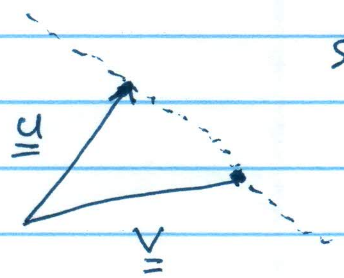
Likewise,

$$\vec{SP} = \tfrac{1}{2}\vec{DB}$$

$$\vec{RQ} = \tfrac{1}{2}\vec{DB}$$

So $\vec{SP}$ and $\vec{RQ}$ are parallel.

Thus $SRQP$ is a parallelogram.

9. a) 

$s\underline{u} + t\underline{v}$ for $s + t = 1$ is the line connecting the endpoints of $\underline{u}, \underline{v}$.

When $s > 0$, $t \geq 0$ then you get the segment between the endpoints.

b) $r\underline{u} + s\underline{v} + t\underline{w}$ with $s + t + u = 1$ describes the plane through the endpoints of $\underline{u}, \underline{v}, \underline{w}$. When $s \geq 0$, $t \geq 0$, $r \geq 0$ we get the solid triangle determined by the three endpoints.

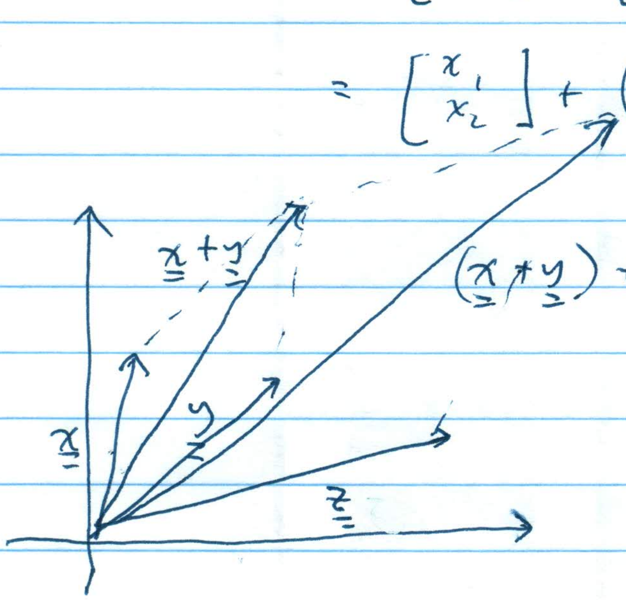
12 b) $(\underline{\underline{x}} + \underline{\underline{y}}) + \underline{\underline{z}} = \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right) + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$= \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \end{bmatrix}$$

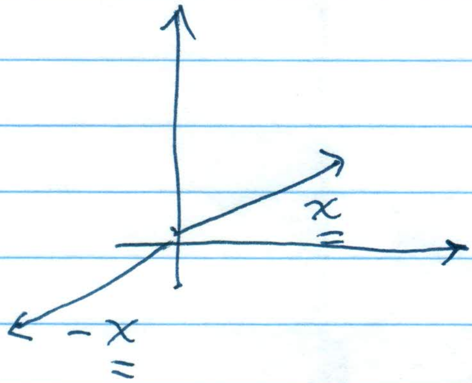
$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \right).$$



$(\underline{\underline{x}} + \underline{\underline{y}}) + \underline{\underline{z}} = \underline{\underline{x}} + (\underline{\underline{y}} + \underline{\underline{z}})$

d) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$



Project:

(5)

Notes:

f) $c(\underline{x} + \underline{y}) = c\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right)$

$= c\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} cx_1 + cy_1 \\ cx_2 + cy_2 \end{bmatrix}$

$= \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix} + \begin{bmatrix} cy_1 \\ cy_2 \end{bmatrix}$

$= c\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c\underline{x} + c\underline{y}$

§ 1.2

1.b) $\underline{x} \cdot \underline{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2 + 1 = -1$

$\cos\theta = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \|\underline{y}\|} = \frac{-1}{\sqrt{5}\sqrt{2}} = \frac{-1}{\sqrt{10}}$

$\theta = \cos^{-1}(-1/\sqrt{10})$

To Do:

d) $\underline{x} \cdot \underline{y} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = 5 + 4 - 9 = 0$

$\cos\theta = \frac{\underline{x} \cdot \underline{y}}{\|x\| \|y\|} = 0$. So $\theta = \pi/2$.

f) $\underline{x} \cdot \underline{y} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -3 + 0 + 5 = 2$

$\cos\theta = \frac{\underline{x} \cdot \underline{y}}{\|\underline{x}\| \|\underline{y}\|} = \frac{2}{\sqrt{50}\sqrt{2}} = \frac{2}{10} = 1/5$

$\theta = \cos^{-1}(1/5)$.

Project:

6

Notes:

2. b) $P_{\underline{\underline{y}}}\, \underline{\underline{x}} = \underline{\underline{x}} \cdot \underline{\underline{y}} \left(\frac{\underline{\underline{y}}}{\|\underline{\underline{y}}\|^2} \right)$

$= -1 \left(\frac{\langle -1, 1 \rangle}{2} \right) = \langle \frac{1}{2}, -\frac{1}{2} \rangle$

$P_{\underline{\underline{x}}}\, \underline{\underline{y}} = \underline{\underline{y}} \cdot \underline{\underline{x}} \left(\frac{\underline{\underline{x}}}{\|\underline{\underline{x}}\|^2} \right)$

$= -1 \left(\frac{\langle 2, 1 \rangle}{5} \right) = \langle -2/5, -1/5 \rangle.$

d) $P_{\underline{\underline{y}}}\, \underline{\underline{x}} = \underline{\underline{x}} \cdot \underline{\underline{y}} \left(\frac{\underline{\underline{y}}}{\|\underline{\underline{y}}\|^2} \right)$

$= \underline{\underline{0}}$

$P_{\underline{\underline{x}}}\, \underline{\underline{y}} = \underline{\underline{y}} \cdot \underline{\underline{x}} \left(\frac{\underline{\underline{x}}}{\|\underline{\underline{x}}\|^2} \right)$

$= \underline{\underline{0}}$

f) $P_{\underline{\underline{y}}}\, \underline{\underline{x}} = \underline{\underline{x}} \cdot \underline{\underline{y}} \left(\frac{\underline{\underline{y}}}{\|\underline{\underline{y}}\|^2} \right) = 2 \left(\frac{\langle -1, 0, 1 \rangle}{2} \right)$

$= \langle -1, 0, 1 \rangle$

$P_{\underline{\underline{x}}}\, \underline{\underline{y}} = \underline{\underline{y}} \cdot \underline{\underline{x}} \left(\frac{\underline{\underline{x}}}{\|\underline{\underline{x}}\|^2} \right) = 2 \left(\frac{\langle 3, -4, 5 \rangle}{50} \right)$

$= \langle 3/25, -4/25, 1/5 \rangle.$

To Do:

Project:

7

Notes:

5. $(\underline{x} - 3\underline{y}) \cdot (\underline{x} + \underline{y})$

$= \|\underline{x}\|^2 - 3\|\underline{y}\|^2 - 2\,\underline{x} \cdot \underline{y} \qquad (*)$

$\frac{1}{4} = \cos\theta = \frac{\underline{x} \cdot \underline{y}}{\|x\|\,\|y\|} = \frac{\underline{x} \cdot \underline{y}}{2 \cdot 1}$

So $\underline{x} \cdot \underline{y} = \frac{1}{2}$.

$\|\underline{x}\| = 2, \ \|\underline{y}\| = 1$.

Thus $(*) = 4 - 3 - 2 \cdot \frac{1}{2} = 0$.

So $(\underline{x} - 3\underline{y}) \perp (\underline{x} + \underline{y})$.

7. $\cos\theta_1 = \frac{\underline{x} \cdot \underline{e}_1}{\|x\|\,\|e_1\|} = \frac{x_1}{\|\underline{x}\|}$

$\cos\theta_2 = \frac{x_2}{\|\underline{x}\|}$

$\cos\theta_3 = \frac{x_3}{\|\underline{x}\|}$

Thus $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = \frac{x_1^2}{\|\underline{x}\|^2} + \frac{x_2^2}{\|\underline{x}\|^2} + \frac{x_3^2}{\|\underline{x}\|^2}$

$= \frac{x_1^2 + x_2^2 + x_3^2}{x_1^2 + x_2^2 + x_3^2} = 1$.

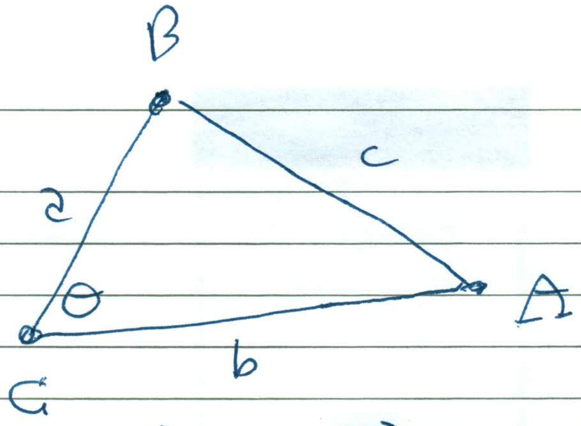
To Do:

Project:

Notes:

To Do:

12.



$$\vec{AB} = \vec{CA} - \vec{CB}$$

So $c^2 = \|\vec{AB}\|^2 = \|\vec{CA} - \vec{CB}\|^2$

$= \|\vec{CA}\|^2 + \|\vec{CB}\|^2 - 2\,\vec{CA}\cdot\vec{CB}$

$= b^2 + a^2 - 2\|\vec{CA}\|\,\|\vec{CB}\|\cos\theta$

$= b^2 + a^2 - 2ab\cos\theta.$

15. a) If $\underline{x}\cdot\underline{y} = 0 \;\; \forall \underline{x} \in \mathbb{R}^n$ then

$\underline{y}\cdot\underline{y} = 0$ so $y_1^2 + y_2^2 + \dots y_n^2 = 0$

hence $y_1 = y_2 = \dots = y_n = 0.$

b) If $\underline{x}\cdot\underline{y} = \underline{x}\cdot\underline{z} \;\; \forall \underline{x} \in \mathbb{R}^n$ then

$\underline{x}\cdot(\underline{y} - \underline{z}) = 0 \;\; \forall x \in \mathbb{R}^n.$

By part (a), $\underline{y} - \underline{z} = 0$ or $\underline{y} = \underline{z}.$

Project:

§ 1,3

Notes:

1,b) Clearly

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0+0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} + \begin{bmatrix} a' \\ b' \\ a'+b' \end{bmatrix} = \begin{bmatrix} a+a' \\ b+b' \\ (a+a')+(b+b') \end{bmatrix}$$

$$c \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = \begin{bmatrix} ca \\ cb \\ ca+cb \end{bmatrix}.$$

So this <u>is</u> a subspace.

d) <u>Not</u> a subspace.

$\underline{x} = \langle 0, 1, 0 \rangle$ is in the space

but $2\underline{x} = \langle 0, 2, 0 \rangle$ is not.

To Do:

f) This is the empty set. It is <u>not</u> a subspace.

Project:

Notes:

To Do:

3. $\underline{\underline{x}} \cdot (c_1 \underline{\underline{v}}_1 + c_2 \underline{\underline{v}}_2 + \cdots + c_k \underline{\underline{v}}_k)$

$= c_1 \underline{\underline{x}} \cdot \underline{\underline{v}}_1 + c_2 \underline{\underline{x}} \cdot \underline{\underline{v}}_2 + \cdots + c_k \underline{\underline{x}} \cdot \underline{\underline{v}}_k$

$= c_1 0 + c_2 0 + \cdots + c_k 0 = 0.$

6. $0 \in U, 0 \in V \Rightarrow 0 \in U \cap V.$

If $\underline{\underline{u}} \in U \cap V, \underline{\underline{v}} \in U \cap V$ then

$\underline{\underline{u}} + \underline{\underline{v}} \in U, \underline{\underline{u}} + \underline{\underline{v}} \in V$ hence $\underline{\underline{u}} + \underline{\underline{v}} \in U \cap V.$

If $\underline{\underline{u}} \in U \cap V$ and $c \in \mathbb{R}$ then

$c\underline{\underline{u}} \in U, c\underline{\underline{u}} \in V$ hence $c\underline{\underline{u}} \in U \cap V.$

Example: $U = \{x \in \mathbb{R}^3 : x + 2y + 3z = 0\}.$

$V = \{x \in \mathbb{R}^3 : 3x + 2y + z = 0\},$

The intersection is a line. Let $x = t$. Then

$$\begin{array}{r} t + 2y + 3z = 0 \\ 3t + 2y + z = 0 \\ \hline -2t \quad + 2z = 0 \\ z = t \end{array}$$

So $y = -2t$. Thus the line is

$x = t, y = -2t, z = t$

Project:

Notes:

To Do:

Example: In the plane, let

$$U = \{(x,y) : y = 2x\}$$

$$V = \{(x,y) : y = 3x\}.$$

The intersection is $\{0\}$, which is a subspace.

9. Let $\underline{x} \in V^{\perp}$. So $\underline{x} \cdot \underline{v} = 0 \quad \forall\, \underline{v} \in V$.

But $U \subset V$ so $\underline{x} \cdot \underline{u} = 0$ for all $\underline{u} \in U$.

So $\underline{x} \in U^{\perp}$. Hence $V^{\perp} \subset U^{\perp}$.

12. Let $\underline{x} \in (U+V)^{\perp}$. Taking $0 \in U$, $\underline{v} \in V$, we see that $x \in V^{\perp}$. Likewise $x \in U^{\perp}$.

So $x \in U^{\perp} \cap V^{\perp}$.

Conversely, if $x \in U^{\perp} \cap V^{\perp}$, let $\underline{u} + \underline{v} \in U + V$. Then

$$\underline{x} \cdot (\underline{u} + \underline{v}) = \underline{x} \cdot \underline{u} + \underline{x} \cdot \underline{v} = 0 + 0 = 0.$$

So $\underline{x} \in (U+V)^{\perp}$