Project:

Math 318

Homework 8 Solutions

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Notes:

Section 4.3

1. a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Clearly the span is 2-dimensional.
The vectors are linearly dependent. Indeed,

$$-2\underline{v}_1 + 0\,\underline{v}_2 + 1\underline{v}_3 = 0.$$

b) $\underline{v}_1$ can be written as a lin. combo. of $\underline{v}_3$.
$\underline{v}_3$ can be written as a lin. combo. of $\underline{v}_1$.
But $\underline{v}_2$ <u>cannot</u> be written as a lin. combo of
$\underline{v}_1, \underline{v}_3$.

To Do:

2. b) These vectors are not multiples of each other, so they are linearly independent.

d) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Clearly the vectors are lin. indep.

Project:

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Notes:

3. If $a(\underline{v}-\underline{w}) + b(2\underline{v}+\underline{w}) = 0$

then $(a+2b)\underline{v} + (-a+b)\underline{w} = 0$.

So $\underline{v}, \underline{w}$ would be linearly dependent, which they are not.

5. If $a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_k \underline{v}_k = \underline{0}$

then compute

$$\underline{v}_j \cdot (a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_k \underline{v}_k) = \underline{v}_j \cdot \underline{0}$$

so

$$a_j \|v_j\|^2 = 0$$

If $\underline{v}_j \neq 0$ we conclude that $a_j = 0$, each j.

So $\underline{v}_1, \dots, \underline{v}_k$ are lin. indep.

To Do:

12. b) Four vectors is too many for a basis of $\mathbb{R}^3$.

d) $\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 2 & -2 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -3 & -4 \end{bmatrix} \to$

Project:

(3)

Notes:

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

So clearly linearly independent. A lin. indep. set of four vectors in $\mathbb{R}^4$ forms a basis.

14. b) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 3 & -1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 1/2 \end{bmatrix}$

Clearly linearly independent.

$$a\begin{pmatrix}1\\0\\3\end{pmatrix} + b\begin{pmatrix}1\\2\\2\end{pmatrix} + c\begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\1\\2\end{pmatrix}$$

To Do:

(1) $a + b + c = 1$

(2) $0a + 2b + 3c = 1$

(3) $3a + 2b + 2c = 2$

$(1) + (2)$ gives $-2a + 0b + c = -1 \Rightarrow a = 0,\ c = -1$

$(2) + (3)$ gives $3a + 0b - c = 1 \quad b = 2$

Project:

Notes:

$\therefore \; 0\underline{v}_1 + 2\underline{v}_2 - 1\underline{v}_3 = \underline{b}$.

d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

So clearly linearly independent.

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + b\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + c\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + d\begin{pmatrix}1\\1\\3\\4\end{pmatrix} = \begin{pmatrix}2\\0\\1\\1\end{pmatrix}$$

To Do:

(1) $a + b + c + d = 2$

(2) $0a + b + c + d = 0$

(3) $0a + 0b + c + 3d = 1$

(4) $0a + 0b + c + 4d = 1$

$(3) + (4) \Rightarrow d = 0, \; c = 1$

$(2) \Rightarrow b = -1$

$(1) \Rightarrow a = 2$

Project:

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Notes:

Hence $2\underline{\underline{v}}_1 + (-1)\underline{\underline{v}}_2 + 1\underline{\underline{v}}_3 + 0\underline{\underline{v}}_4 = \underline{\underline{b}}$.

Section 5.1.

1 b) This set is bounded by 1 and given by a non-strict inequality hence closed. So compact.

d) This set is unbounded as (k, k) is in the set for every integer k, So not compact.

f) The set is unbounded because $e^t \to +\infty$. So not compact.

2. If X is not closed, then let $\underline{\underline{a}}$ be a limit point that is not in X. Define $f(\underline{\underline{x}}) = \frac{1}{\|\underline{\underline{x}} - \underline{\underline{a}}\|}$. This is continuous and unbounded.

If X is not bounded, then let $f(\underline{\underline{x}}) = \|\underline{\underline{x}}\|$. This continuous and unbounded.

4. b) Need to calculate the maximum of

$$\left\| \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \right\|$$

To Do:

Project:

Notes:

This is

$$\left\| \begin{pmatrix} 3\cos\theta + 4\sin\theta \\ 3\cdot\cos\theta + 4\sin\theta \end{pmatrix} \right\|$$

$$= \sqrt{(3\cos\theta + 4\sin\theta)^2 + (3\cos\theta + 4\sin\theta)^2}$$

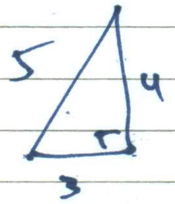
$$= \sqrt{2}\,|3\cos\theta + 4\sin\theta|$$

Let $\varphi(\theta) = 3\cos\theta + 4\sin\theta$

so $\varphi'(\theta) = -3\sin\theta + 4\cos\theta = 0$

when $\frac{3}{4}\tan\theta = 1$

or $\theta = \tan^{-1}\frac{4}{3}$.



$\cos\tan^{-1}\frac{4}{3} = \frac{3}{5}$

$\sin\tan^{-1}\frac{4}{3} = \frac{4}{5}$.

So the max is

$$\left\| \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix} \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right\| = \sqrt{50} = 5\sqrt{2}.$$

To Do:

Project:

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Notes:

8. Let $f : S \to \mathbb{R}$ be given by

$$f(\underline{x}) = \|\underline{x} - \underline{a}\|.$$

Then f is a continuous function on a compact set. So there is a point $\underline{x}_0$ in S at which f attains its minimum value.

11. Let $\underline{e}_1 \in S_1$, $\underline{e}_2 \in S_2$, ...
Then $\{\underline{e}_j\}$ is a sequence in S_1 so there is a convergent subsequence $\underline{e}_{j_k} \to \underline{x}$.
And $\underline{x} \in S_1$. But the subsequence also converges in S_2. So $\underline{x} \in S_2$. Likewise $\underline{x} \in S_j \ \forall j$.

To Do:

Section 5.2

1. b) $Df = (y+1 \quad x-1)$

$$\begin{aligned} y+1 &= 0 \\ x-1 &= 0 \end{aligned} \Rightarrow (1,-1) \text{ is critical pt.}$$

Project:

f) $Df = (2xe^{-y} \quad -x^2e^{-y} + 2ye^{-y} - y^2e^{-y})$

$$2xe^{-y} = 0 \Rightarrow x = 0$$

$$-x^2e^{-y} + 2ye^{-y} - y^2e^{-y} = 0 \Rightarrow 2ye^{-y} - y^2e^{-y} = 0$$

$$ye^{-y}(2-y) = 0 \Rightarrow y = 0 \text{ or } 2$$

Critical points are $(0,0)$, $(0,2)$,

Notes:

h) $Df = (2xy - 4y \quad x^2 - 4x - 2y)$

$$2xy - 4y = 0 \Rightarrow 2y(x-2) = 0 \begin{cases} y = 0 \\ x = 2 \end{cases}$$

$$x^2 - 4x - 2y = 0 \quad (*)$$

If $y = 0$ then $(*)$ gives $x^2 - 4x = 0 \Rightarrow \begin{matrix} x = 4 \\ x = 0 \end{matrix}$

If $x = 2$ then $(*)$ gives $4 - 8 - 2y = 0 \Rightarrow y = -2$

Critical points are $(4,0)$, $(0,0)$, $(2,-2)$.

To Do:

2. If the box has width x and length y then the height is $z = \dfrac{6-x-2y}{3}$.

So the volume is

$$V = x \cdot y \cdot \frac{6-x-2y}{3}.$$

$$= 2xy - \frac{x^2y}{3} - \frac{2xy^2}{3}$$

Project:

Project:

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Notes:

$$DV = \left[2y - \frac{2xy}{3} - \frac{2y^2}{3} \quad 2x - \frac{x^2}{3} - \frac{4xy}{3}\right]$$

(1) $2y - \frac{2xy}{3} - \frac{2y^2}{3} = 0$

(2) $2x - \frac{x^2}{3} - \frac{4xy}{3} = 0$

By (1), $-\frac{2}{3}y^2 + \left(2 - \frac{2x}{3}\right)y = 0$

$$y\left(-\frac{2}{3}y + \left(2 - \frac{2}{3}x\right)\right) = 0$$

So $y = 0$ or $y = 3 - x$.

If $y = 0$ then (2) gives $2x - \frac{x^2}{3} = 0$

$$x\left(2 - \frac{x}{3}\right) = 0 \Rightarrow x = 0, 6.$$

If $y = 3 - x$ then (2) gives

$$2x - \frac{x^2}{3} - \frac{4x(3-x)}{3} = 0$$

$$2x - \frac{x^2}{3} - 4x + \frac{4}{3}x^2 = 0$$

$$x^2 - 2x = 0 \Rightarrow x = 0 \text{ or } 2$$

So the critical points are $(0,0), (6,0)$, $(0,3), (2,1)$. Only $(2,1)$ makes sense.

To Do:

Project:

Notes:

To Do:

4. $Df = (2x-2 \quad 4y)$

$2x-2=0 \qquad x=2$

$4y=0. \qquad y=0$

So $(2,0)$ is a critical point but it's not in the domain of the fcn.

So the extrema are on the edge.

Set $x=\sqrt{2}\cos\theta$, $y=\sqrt{2}\sin\theta$. We must find extrema for

$$g(\theta) = (\sqrt{2}\cos\theta)^2 + 2(\sqrt{2}\sin\theta)^2 - 2\sqrt{2}\cos\theta$$

$$= 2\cos^2\theta + 4\sin^2\theta - 2\sqrt{2}\cos\theta$$

$$g'(\theta) = -4\cos\theta\sin\theta + 8\sin\theta\cos\theta + 2\sqrt{2}\sin\theta$$

$$= 4\cos\theta\sin\theta + 2\sqrt{2}\sin\theta = 0$$

if $2\sin\theta\,[2\cos\theta + \sqrt{2}] = 0$

so $\sin\theta = 0 \Rightarrow \theta = 0, \pi$

or $2\cos\theta + \sqrt{2} = 0 \Rightarrow \theta = \frac{3\pi}{4}, \frac{5\pi}{4}$.

$g(0) = 2 + 0 - 2\sqrt{2} = 2(1-\sqrt{2})$

$g(\pi) = 2 + 0 + 2\sqrt{2} = 2(1+\sqrt{2})$

$g(\frac{3\pi}{4}) = 1 + 2 + 2 = 5$

$g(\frac{5\pi}{4}) = 1 + 2 + 2 = 5$.

max $= 5$, min $= 2(1-\sqrt{2})$.

Project:

Notes:

9. Let the x intercept be a and the y intercept be b.

Then the base is bounded by $y = b - \frac{b}{a}x$.

The plane is

$$\alpha x + \beta y + \gamma z = 1$$

$$\alpha a = 1$$

$$\beta b = 1$$

$$\alpha 1 + \beta 2 + \gamma 2 = 1$$

$$\gamma = \frac{1}{2} - \frac{1}{2a} - \frac{1}{b}.$$

So the plane is

$$\frac{1}{a}x + \frac{1}{b}y + \left(\frac{1}{2} - \frac{1}{2a} - \frac{1}{b}\right)z = 1$$

or $$z = \frac{1 - x/a - y/b}{\frac{1}{2} - \frac{1}{2a} - \frac{1}{b}} = \frac{2ab - 2bx - 2ay}{ab - b - 2a}.$$

The volume is

$$V(a,b) = \int_0^a \int_0^{b - \frac{b}{a}x} \int_0^{\frac{2ab - 2bx - 2ay}{ab - b - 2a}} 1 \, dz \, dy \, dx$$

$$= \int_0^a \int_0^{b - \frac{b}{a}x} \frac{2ab - 2bx - 2ay}{ab - b - 2a} \, dy \, dx$$

To Do:

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Notes:

$$= \int_0^a \left[\frac{2aby}{ab-b-2a} - \frac{2bxy}{ab-b-2a} - \frac{ay^2}{ab-b-2a} \right]_0^{y = b - \frac{b}{a}x} dx$$

$$= \int_0^a \frac{2ab\left(b-\frac{b}{a}x\right)}{ab-b-2a} - \frac{2bx\left(b-\frac{b}{a}x\right)}{ab-b-2a} - \frac{a\left(b-\frac{b}{a}x\right)^2}{ab-b-2a} \, dx$$

$$= \int_0^a \frac{2ab^2 - 2b^2x}{ab-b-2a} + \frac{-2b^2x + \frac{2b^2}{a}x^2}{ab-b-2a}$$

$$+ \frac{-ab^2 + 2b^2x - \frac{b^2x^2}{a}}{ab-b-2a} \, dx$$

$$= \frac{2ab^2x - b^2x^2}{ab-b-2a} + \frac{-b^2x^2 + \frac{2}{3}\frac{b^2}{a}x^3}{ab-b-2a}$$

$$+ \left. \frac{-ab^2x + b^2x^2 - \frac{b^2x^3}{3a}}{ab-b-2a} \right]_0^a$$

$$= \frac{2a^2b^2 - b^2a^2}{ab-b-2a} + \frac{-b^2a^2 + \frac{2}{3}a^2b^2}{ab-b-2a}$$

$$+ \frac{-a^2b^2 + b^2a^2 - \frac{b^2a^2}{3}}{ab-b-2a}$$

$$= \frac{\frac{1}{3}b^2a^2}{ab-b-2a}$$

To Do:

Project:

I 2b

Notes:

$$DV = \left(\frac{(ab-b-2a)\frac{2}{3}b^2a - \frac{1}{3}b^2a^2(b-2)}{(ab-b-2a)^2}, \frac{(ab-b-2a)\frac{2}{3}ba^2 - \frac{1}{3}b^2a^2(a-1)}{(ab-b-2a)^2} \right)$$

$$= \left(\frac{\frac{2}{3}b^3a^2 - \frac{2}{3}b^3a - \frac{4}{3}b^2a^2 - \frac{1}{3}b^3a^2 + \frac{2}{3}b^2a^2}{(ab-b-2a)^2}, \frac{\frac{2}{3}a^3b^2 - \frac{2}{3}b^2a^2 - \frac{4}{3}ba^3 - \frac{1}{3}b^2a^3 + \frac{1}{3}b^2a^2}{(ab-b-2a)^2} \right)$$

$$= \left(\frac{\frac{1}{3}b^3a^2 - \frac{2}{3}b^3a - \frac{2}{3}b^2a^2}{(ab-b-2a)^2} \quad \frac{\frac{1}{3}b^2a^3 - \frac{1}{3}b^2a^2 - \frac{4}{3}ba^3}{(ab-b-2a)^2} \right.$$

$$\frac{1}{3}b^3a^2 - \frac{2}{3}b^3a - \frac{2}{3}b^2a^2 = 0$$

$$\frac{1}{3}b^2a^3 - \frac{1}{3}b^2a^2 - \frac{4}{3}ba^3 = 0$$

$$\frac{b^2a}{3}\left[ba - 2b - 2a\right] = 0$$

$$\frac{ba^2}{3}\left[ba - b - 4a\right] = 0$$

So $b=0$ or $a=0$ (of no interest) or

$$ba - 2b - 2a = 0$$

$$ba - b - 4a = 0$$

To Do:

Project:

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Notes:

Thus $2b + 2a = b + 4a$

$$b - 2a = 0$$

$$b = 2a.$$

Hence $2a^2 - 4a - 2a = 0.$

$$2a(a-3) = 0$$

$$a = 0 \text{ or } a = 3.$$

Clearly, $a = 3$, $b = 6$ is the solution we seek.

To Do:

Project:

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Section 5.3

Notes:

1. b) The crit. pt was $(1,-1)$

$$Hf = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The quadratic form is

$$1h_1h_2 + 1h_1h_2 = 2h_1h_2.$$

This is indefinite.

f) Critical points are $(0,0)$, $(0,2)$

$$Hf = \begin{pmatrix} 2e^{-y} & -2xe^{-y} \\ -2xe^{-y} & x^2e^{-y} + 2e^{-y} - 2ye^{-y} - 2ye^{-y} + y^2e^{-y} \end{pmatrix}$$

$$Hf\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad q_\ell = 2h_1^2 + 2h_2^2$$

$(0,0)$ is local min.

$$Hf\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2e^{-2} & 0 \\ 0 & -2e^{-2} \end{pmatrix} \quad q_\ell = 2h_1^2 - 2h_2^2.$$

saddle.

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Notes:

h) ~~d)~~ Critical points are $(4,0), (0,0), (2,-2)$

$$H = \begin{pmatrix} 2y & 2x-4 \\ 2x-4 & -2 \end{pmatrix}$$

$$\mathcal{H}\begin{pmatrix}4\\0\end{pmatrix} = \underline{h}^T\begin{pmatrix} 0 & 4 \\ 4 & -2 \end{pmatrix}\underline{h} = 8h_1h_2 - 2h_2^2$$

indefinite = saddle

$$\mathcal{H}\begin{pmatrix}0\\0\end{pmatrix} = \underline{h}^T\begin{pmatrix} 0 & -4 \\ -4 & -2 \end{pmatrix}\underline{h} = -8h_1h_2 - 2h_2^2$$

indefinite = saddle

$$\mathcal{H}\begin{pmatrix}2\\-2\end{pmatrix} = \underline{h}^T\begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}\underline{h} = -4h_1^2 - 2h_2^2$$

neg. def

= local max.

To Do:

4. $Df = (3x^2 - 3e^y \quad 3e^{3y} - 3xe^y)$

$$3x^2 - 3e^y = 0$$
$$3e^{3y} - 3xe^y = 0$$

crit pt. = $(1,0)$

$$Hf = \begin{pmatrix} 6x & -3e^y \\ -3e^y & 9e^{3y} - 3xe^y \end{pmatrix}$$

$$\mathcal{H}\begin{pmatrix}1\\0\end{pmatrix} = \underline{h}^T\begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}\underline{h} = 6h_1^2 + 6h_2^2 - 6h_1h_2$$

$$= 6(h_1 - h_2)^2 + 6h_1h_2$$

$$\geq 6\left[h_1^2 + h_2^2 - h_1h_2\right]$$

$$\geq 6\left[h_1^2 + h_2^2 - \frac{h_1^2}{2} - \frac{h_2^2}{2}\right] = 3(h_1^2 + h_2^2)$$

So $(1,0)$ is clearly a local minimum.

It is not a global minimum

because $f(1,0) = -1$

while $f(-3,0) = -27 + 1 + 9 = -17$,

5. If $\frac{\partial^2 f}{\partial x^2}(\underline{a}) \neq 0$ then $\frac{\partial^2 f}{\partial y^2}(\underline{a}) = -\frac{\partial^2 f}{\partial y^2}(\underline{a})$

and the Hessian at $\underline{a}$ can be neither

positive definite nor negative definite.