Math 318 Spring, 2013
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Second Midterm

General Instructions: Read the statement of each problem carefully. Do only what is requested—nothing more and nothing less. Provide a *complete solution* to each problem.

If you only write the answer then you will not get full credit. If you need extra room for your work then use the backs of the pages.

Be sure to ask questions if anything is unclear.

(10 points) **1.** Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} : U \to \mathbb{R}^m$ be a function. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Let $\mathbf{a} \in \mathbb{R}^n$. Give a rigorous definition of the directional derivative $D_{\mathbf{v}} f(\mathbf{a})$.

$$D_{\mathbf{v}} \mathbf{f}(\mathbf{a}) = \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{a} + t\mathbf{v}) - \mathbf{f}(\mathbf{a})}{t} .$$

(10 points) **2.** Let $U \subseteq \mathbb{R}^n$ be open and $\mathbf{a} \in U$. Let $\mathbf{f} : U \to \mathbb{R}^m$ be a function. Give

the precise definition of what it means for **f** to be differentiable at **a**.

$\mathbf{f}$ is differentiable at $\mathbf{a}$ if there is a linear map $D\mathbf{f}(\mathbf{a})$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ such that

$$\lim_{\mathbf{h}\to 0} \frac{\mathbf{f}(\mathbf{a}+\mathbf{h}) - \mathbf{f}(\mathbf{a}) - D\mathbf{f}(\mathbf{a})\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}.$$

(10 points) **3.** Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if} \quad (x,y) \neq 0\,, \\ 0 & \text{if} \quad (x,y) = 0 \end{cases}$$

is *not* differentiable at the origin in $\mathbb{R}^2$.

If it were differentiable at 0, then the entries of Df would be the partial derivatives. These are 0. So the question is whether

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{0}+\mathbf{h}) - f(\mathbf{0}) - \mathbf{0}}{\|\mathbf{h}\|} = \mathbf{0} \quad \text{or} \quad \lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

If we examine this limit on the line $h=k$ we find

$$\lim_{h\to 0} \frac{h^2/(h^2+h^2)}{|h|} \to \infty.$$ So the limit is not 0 and

f is not differentiable at $\mathbf{0}$.

(10 points) **4.** Define

$$\mathbf{f}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - x \\ e^{xy} \\ xy - y^2 \end{pmatrix}$$

and

$$\mathbf{g}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - 3y + 2z \\ x + y + z \end{pmatrix}.$$

Use the Chain Rule to calculate $D(\mathbf{f} \circ \mathbf{g})(\mathbf{0})$ and $D(\mathbf{g} \circ \mathbf{f})(\mathbf{0})$.

$D(\underline{f} \circ \underline{g})(\underline{0}) = D\underline{f}(\underline{g}(\underline{0})) \cdot D\underline{g}(\underline{0})$. $\quad D\underline{f}(\underline{0}) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$D(\underline{g} \circ \underline{f})(\underline{0}) = D\underline{g}(\underline{f}(\underline{0})) \cdot D\underline{f}(\underline{0}) \quad D\underline{g}(0) = \begin{pmatrix} 1 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$D\underline{f} = \begin{pmatrix} -1 & 1 \\ ye^{xy} & xe^{xy} \\ y & x-2y \end{pmatrix} \quad D\underline{f}(\underline{g}(\underline{0})) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad D\underline{g}(\underline{f}(\underline{0})) = \begin{pmatrix} 1 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$D(\underline{f} \circ \underline{g})(\underline{0}) = D\underline{f}(\underline{g}(\underline{0}))\, D\underline{g}(\underline{0}) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$D(\underline{g} \circ \underline{f})(\underline{0}) = D\underline{g}(\underline{f}(\underline{0}))\, D\underline{f}(\underline{0}) = \begin{pmatrix} 1 & -3 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

(10 points) **5.** Write down (but do not evaluate) the integral that represents the arc length of the curve

$$\gamma(t) = \left(e^t, \sin 2t, t^2\right)$$

$$\ell = \int_1^4 \|\gamma'(t)\|\, dt = \int_1^4 \|(e^t, -2\cos 2t, 2t)\|\, dt$$

$$= \int_1^4 \sqrt{e^{2t} + 4\cos^2 2t + 4t^2}\, dt.$$

between $t = 1$ and $t = 4$.

(10 points) **6.** Use Gaussian elimination to completely solve the linear system

$$\begin{array}{cccccc} 2x_1 & & -x_3 & +x_4 & = & 1 \\ x_1 & +x_2 & -x_3 & +2x_4 & = & 0 \\ & 2x_2 & +x_3 & -x_4 & = & 2 \end{array}$$

$$\left(\begin{array}{cccc|c} 2 & 0 & -1 & 1 & 1 \\ 1 & 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 1/2 \\ 1 & 1 & -1 & 2 & 0 \\ 0 & 2 & 1 & -1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 3/2 & -1/2 \\ 0 & 2 & 1 & -1 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 3/2 & -1/2 \\ 0 & 0 & 2 & -4 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/2 & 5/4 \\ 0 & 1 & 0 & 1/2 & 1/4 \\ 0 & 0 & 2 & -4 & 3 \end{array}\right)$$

$x_4 = t$

$x_3 = 2t + 3/2$

$x_2 = -\frac{1}{2}t + \frac{1}{4}$

$x_1 = \frac{1}{2}t + \frac{5}{4}$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = t \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 5/4 \\ 1/4 \\ 3/2 \\ 0 \end{pmatrix}.$$ This is a line.

(10 points) **7.** Use Gaussian elimination to find the inverse of the matrix

$$\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1/3 & -1/3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1/3 & -1/3 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1/3 & -1/3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & -3 \end{array}\right]$$

The inverse of the given matrix is

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 3 & -3 \end{bmatrix}$$

(10 points) **8.** Calculate the curvature κ of the curve $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ given by

$$\mathbf{g}(t) = \begin{pmatrix} \cos 2t \\ \sin 2t \\ 2t \end{pmatrix}$$

at the point $\mathbf{g}(0) = (1, 0, 0)$.

Reparametrize the curve as

$$\tilde{g}(s) = \begin{pmatrix} \cos s/\sqrt{2} \\ \sin s/\sqrt{2} \\ s/\sqrt{2} \end{pmatrix}. \text{ Then } \tilde{g}'(s) = \begin{pmatrix} -\frac{1}{\sqrt{2}} \sin s/\sqrt{2} \\ 1/\sqrt{2} \cos s/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

and $\|\tilde{g}'(s)\| = \sqrt{\frac{1}{2}\sin^2 s/_2 + \frac{1}{2}\cos^2 s/_2 + 1/_2} = 1$. So $\tilde{g}$ is parametrized acc. to arc length. Then

$$\kappa(s) = \|T'(s)\| = \left\| \begin{pmatrix} -\frac{1}{2}\cos s/\sqrt{2} \\ -1/2 \sin s/\sqrt{2} \\ 0 \end{pmatrix} \right\| = \sqrt{\frac{1}{4}\cos^2 \frac{s}{\sqrt{2}} + \frac{1}{4}\sin^2 s/\sqrt{2}} = \frac{1}{2}.$$

(10 points) **9.** At the point $(1, 1)$, what is the direction of greatest increase of the function $f(x, y) = x^2 + y^2$? Give a rigorous mathematical explanation for your answer.

The "direction" of greatest increase is

$$\nabla f(1, 1) = \langle 2x, 2y \rangle \big|_{(1,1)} = \langle 2, 2 \rangle.$$

The actual unit vector direction $\underline{v}$ is then

$$\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

(10 points) **10.** What is the equation of the tangent plane to the graph of $f(x, y) = x^2 - y^2$ at the point $(1, 1, 0)$?

The plane passes through $(1, 1, 0)$ and has normal vector

$$\langle \nabla f(x,y), -1 \rangle$$

$$= \langle 2x, -2y, -1 \rangle \Big|_{(1,1,0)} = \langle 2, -2, -1 \rangle.$$

Thus the equation is

$$\langle 2, -2, -1 \rangle \cdot \langle x-1, y-1, z-0 \rangle = 0$$

$2x - 2y - 1z = 0$. This is the tangent plane.