MATH 470

SOLUTIONS TO FINAL EXAM

1. Consider K_n, $n \geq 3$. There are $\binom{n}{k}$ ways of choosing k vertices from n. There are $k!$ ways of ordering the vertices to form a cycle.

We want distinct cycles, and each cycle occurs here k times (i.e., in the same order but beginning at a different vertex). So we have

$$\frac{1}{k}\binom{n}{k}\cdot k! = \binom{n}{k}\cdot(k-1)!$$

cycles on k vertices. The total number of cycles is

$$\sum_{k=3}^{n}\binom{n}{k}\cdot(k-1)!$$

2. Let G be a given graph. Let G_1, G_2 be even subgraphs. Let $v \in V(G)$. Let S_1 be the set of incident edges to v in G_1 and S_2 be the set of incident edges to v in G_2.

Then $|S_1 \Delta S_2| = |S_1 \setminus (S_1 \cap S_2)| + |S_2 \setminus (S_1 \cap S_2)|$.

Since G_1, G_2 are even subgraphs, $|S_1|$ and $|S_2|$

are both even. So

$$|S_1 \setminus (S_1 \cap S_2)| \text{ and } |S_2 \setminus (S_1 \cap S_2)|$$

have the same parity. Hence $|S_1 \Delta S_2|$ is even. Since this assertion holds for every vertex v, $G_1 \Delta G_2$ is even.

In case G_1, G_2 are both odd, similar reasoning shows that $|S_1 \setminus (S_1 \cap S_2)|$ and $|S_2 \setminus (S_1 \cap S_2)|$ have the same parity. So $|S_1 \Delta S_2|$ is even and $G_1 \Delta G_2$ is even.

3. There are many different ways to answer this question:

a) $K_{3,3}$ has six vertices and $K_{3,2} + K_3$ has eight vertices. So $K_{3,3}$ has characteristic polynomial of degree 6 while $K_{3,2} + K_3$ has characteristic polynomial of degree 8. The characteristic polynomials are unequal hence the graphs are <u>not</u> isomorphic.

b) The adjacency matrix for $K_{3,3}$ has eigenvalues $3, -3,$

The adjacency matrix for $K_{2,3} + K_3$ has eigenvalues $3, \sqrt{6}, -\sqrt{6},$

Since the eigenvalues are different, the graphs are not isomorphic.

4. The degree of the characteristic polynomial tells us the number of vertices, so $n(G) = 8$. The coefficient of $\lambda^{n(G)-2} = \lambda^6$ equals $-e(G)$, so $e(G) = 24$.

Corollary 8.6.6 tells us that the coefficient of $\lambda^{n(G)-3} = \lambda^5$ is -2 times the number of triangles in G. So G has 32 triangles.

Taking all this information into account, we must delete 4 edges from K_8 to achieve 24 edges and 32 triangles. If we do that in a generic way we get the graph

$$K_{2,2,2,2}$$

A calculation verifies that this has the right char. poly.

5. The probability that i is fixed, $1 \le i \le n$, is

$$\frac{(n-1)!}{n!} = \frac{1}{n}.$$

Let X_i be the indicator variable of i being fixed. Then

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} 1 \cdot \frac{1}{n} = 1.$$

6. For a vertex to have degree k, it must have k neighbors. The probability of this happening is

$$\binom{n-1}{k} p^k (1-p)^{n-1-k}.$$

Let X_i be the indicator variable of the i^{th} vertex being of degree k. Then

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \binom{n-1}{k} p^k (1-p)^k$$

$$= n \binom{n-1}{k} p^k (1-p)^k.$$

7. Of course there are $\binom{6}{3}$ triples of vertices. Number these triples t_i, $1 \le i \le \binom{6}{3}$. Let X_i be the indicator variable that the triple t_i is monochromatic. Then

$$E\left(\sum_i X_i\right) = \sum_i E(X_i) = \sum_i \left[1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)\right]$$

$$= \binom{6}{3} P(X_i = 1)$$

$$= \binom{6}{3} \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{1}{4} = 5.$$

8. Let $A \subseteq V(G)$ be a randomly chosen set, so for each vertex $v \in G$, $v \in A$ with probability p and $v \notin A$ with probability $1-p$. Say that the subgraph S has M vertices and N edges. Certainly S has an independent set of size $M-N$. Also G has $\frac{nd}{2}$ edges. So

$$E(M-N) = E(M) - E(N) = pn - p^2 \cdot \frac{nd}{2}$$

$$= -\frac{nd}{2}\left(p - \frac{1}{d}\right)^2 + \frac{n}{2d}.$$

This last expression is maximized (using calculus) when $p = 1/d$. Taking $p = 1/d$, we see that

$$E(M-N) = \frac{n}{2d}.$$

So there is an indep. set with at least $n/(2d)$ vertices.

9. a) Every tree is bipartite. We induct on the number $n(T)$ of vertices.

<u>Basis Step</u>: $\underline{\underline{n=1}}$ A single vertex is trivially bipartite.

<u>Inductive Step</u>: $n \geq 1$ Suppose every tree on n vertices is bipartite. Let T be a tree with $n+1$ vertices. Since T has at least 2 vertices, T has at least two leaves. Let v be one of the leaves. Set $T' = T \setminus v$. Then T' is still a tree, so the inductive hypothesis implies that T' is bipartite. For T, we leave all vertices from T' in their partite sets, and we assign v into the partite set that does <u>not</u> contain its only neighbor. This gives a bipartition for T.

b) Of course $K_{2,3}$ is bipartite, but it is not a tree because it contains cycles.

10. Let T be a tree with average degree a. Certainly

$$a = \frac{\sum_{v \in V(T)} d(v)}{n(T)} = \frac{2e(T)}{n(T)} = \frac{2(n(T)-1)}{n(T)}.$$

Solving for $n(T)$, we see that $n(T) = \frac{2}{2-a}$.

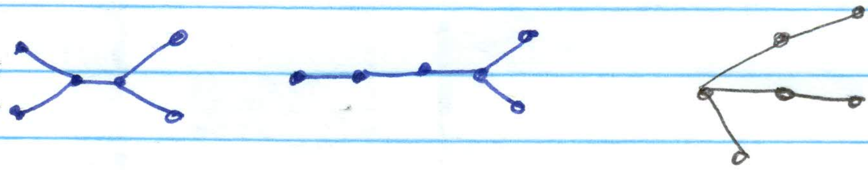
11. Let G be a graph w/ n vertices, m edges, and k components. If we choose a spanning tree from each component, that uses $n-k$ edges. Since adding an edge to a tree forms just one cycle, each of the remaining $m-n+k$ edges completes a cycle w/ edges in the spanning forest. The cycles formed are distinct, because the edges being added are distinct. So G has $m-n+k$ cycles. Since $k \geq 1$, G has at least $m-n+1$ cycles.

12. The number of vertices in a graph is the sum of the number of vertices in each component. Same for edges. So a graph w/ fewer edges than vertices must have a component w/ fewer edges than vertices. Such a component G^* is connected, and has $n(G^*)-1$ edges. So it is a tree.

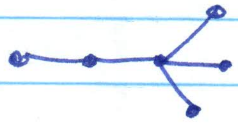
13. An edge is a cut-edge iff it belongs to no cycle. So every edge in G is a cut-edge iff G has no cycle. If G is connected, then G is a tree.

Conversely, if G is a tree, then we know it is connected and every edge is a cut-edge.

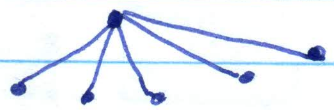
14. $k=3$



$k=4$

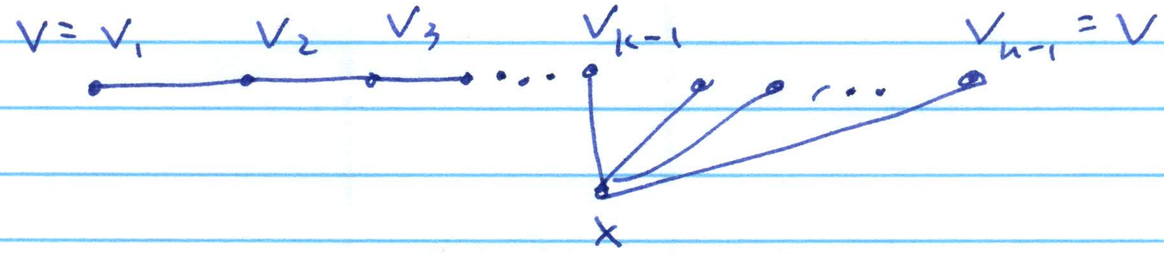


$k=5$



$k=6$ Impossible.

15. Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices of P_{n-1} in order. Let x be the added vertex that is adjacent to each v_j. Then the spanning tree



has diameter k.