Ma 322: Biostatistics
Homework Assignment 4

Prof. Wickerhauser

Read Chapter 9, "An Introduction to Bayesian Data Analysis," pages 135–159 of our text.

Note: There are many errors in the relevant formulas in our textbook. Use the correct formula

$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A).$$

1. Suppose that X is a probability space with probability function P, and $A, B \subset X$ are events satisfying $P(A) = 0.6$ and $P(B) = 0.3$.

   (a) If $P(A|B) = 0.4$, what is $P(B|A)$?

   (b) If $P(A \cap B) = 0.1$, what are $P(A|B)$ and $P(B|A)$?

   (c) If $P(A \cap B) = 0.2$, what are $P(A|B^c)$ and $P(B|A^c)$? (Here A^c is the complement of A, while B^c is the complement of B.)

2. Candidate prior pdfs from the Beta family of densities have the following form:

$$f_{\alpha,\beta}(x) = \frac{1}{\beta(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

   (a) Find the mean and standard deviation of $f_{\alpha,\beta}$ in terms of α and β. (Hint: see p.103 of our textbook.)

   (b) For what values of α, β is $f_{\alpha,\beta}$ the uniform density on $[0, 1]$? Plot this beta pdf to check your result.

   (c) Professor Bayes believes that an experiment has probability $p = 0.3$ of succeeding, but concedes that his p is the mean of a beta density with standard deviation 0.1. To model these beliefs, what values of α, β should Bayes use in the prior pdf $f_{\alpha,\beta}(p)$? Plot your solution to check your work. Evaluate this with the `R` command `dbeta(x,shape1=alpha,shape2=beta)`.

3. Individuals from two subpopulations, A and B, of a certain species of passerines (perching birds) are mixed to form the population under study. They carry a gene mutation with respective incidences $p_A = 0.1$ and $p_B = 0.8$, but are otherwise indistinguishable. The gene expresses a detectable plasma protein marker in individuals that have it. Individuals without the gene mutation do not express the marker protein.

   (a) One individual is tested and found to have the plasma marker. Assuming that A and B are present in equal numbers, compute the probability that the individual comes from population A. Then compute the probability that the individual comes from population B.

   (b) A random sample of 93 individuals is tested and 61 of them have the plasma marker. Use Bayes' rule with the uniform prior $f_{\alpha,\beta}(a) = 1$ on the proportion a of A's; include this information and compute the posterior distribution of the proportion of A's and B's.

4. Given a gamma prior pdf

$$f_{\alpha,\beta}(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$$

   with shape parameters $\alpha = 4$ and $\beta = 2$, and an experiment that produces a count of $x = 19$ which is expected to have the Poisson density

$$P(x|\theta) = \theta^x e^{-\theta}/x!,$$

   find the posterior pdf $P(\theta|x = 19)$ for the parameter θ. Ignore normalization constants, but plot the result for a range of θ's that includes the likeliest.

5. Given a gamma prior pdf

$$f_{\alpha,\beta}(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$$

   with shape parameters $\alpha = 3$ and $\beta = 5$, and an experiment that produces a time-to-failure of $t = 2$ which is expected to have the exponential density

$$P(t|\lambda) = \lambda e^{-\lambda t},$$

   find the posterior pdf $P(\lambda|t = 2)$ for the parameter λ. Ignore normalization constants, but plot the result for a range of λ's that includes the likeliest.