Math 322: Biostatistics Midterm Examination

Name:________________________________

Wednesday, 9 March 2016

7 Problems on 1+7 Pages

You may not use any calculation aids or reference materials. You have 50 minutes. Please write your complete solutions on this exam.

Some formulas:

- Bayes' Theorem: $P(B|A)P(A) = P(A|B)P(B) = P(A \cap B)$.
- Detailed balance equations for a Markov chain transition matrix M with stationary density π:
$$\pi[i]M[i,j] = \pi[j]M[j,i], \qquad \text{for all } i, j.$$
- Maximums: Function $f = f(x)$ has a maximum at $x = x_0$ if and only if $\log f(x)$ has a maximum at $x = x_0$.
- Integrals: For all $n > 1$ and all $y \geq 1$,
$$\int_1^y x^{-n}\,dx = \left.\frac{x^{-n+1}}{-n+1}\right|_1^y = \frac{1}{n-1}\left(1 - \frac{1}{y^{n-1}}\right).$$
- Derivatives: $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}\log x = \frac{1}{x}$, $\frac{d}{dx}\log[x^a(1-x)^b] = \frac{a}{x} - \frac{b}{1-x}$.

1. Suppose that A, B, C, D are nonempty subsets of X satisfying the following conditions:

   - $A \cap B \neq \emptyset$
   - $A \cap C \neq \emptyset$
   - $B \cap C \neq \emptyset$
   - $A \cap B \cap C = \emptyset$
   - $A \subset D$
   - $B \subset D$
   - $C \subset D$

   Graph the sets A, B, C, D, X using a Venn diagram.

2. At noon, Guildenstern rolls a fair die. If it shows 1 or 2 spots, he eats an apple.

   If the die shows 3,4,5, or 6 spots, Guildenstern then flips a fair coin. If the coin shows heads he eats an apple, otherwise he eats a banana.

   (a) What is the conditional probability that Guildenstern eats an apple given that the die showed 3 spots?

   (b) What is the probability that Guildenstern eats an apple? What is the probability that he eats a banana?

   (c) What is the conditional probability that the die showed 3 spots given that Guildenstern eats an apple?

3. A sailing club has nine human members (1,. . . ,9) and four sailboats (A,B,C,D).

   (a) How many distinct two-member crews can be formed?

   (b) In how many distinct ways can all four sailboats be assigned two-member crews?

   Note: A correct formula in terms of binomial coefficients $\binom{n}{k}$ is a sufficient answer.

4. Suppose that the success probability $p \in [0, 1]$ has a beta prior pdf

$$f_{\alpha,\beta}(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

   with shape parameters $\alpha = 3$ and $\beta = 4$. Suppose that an experiment which is expected to have the binomial density

$$P(k|p) = \binom{n}{k} p^k (1-p)^{n-k}$$

produces $k = 4$ successes in $n = 9$ trials.

(a) Find the posterior pdf $P(p|k = 4)$ for the parameter p. You may ignore normalization constants.

(b) From the pdf of part a, find the MLE, namely the most likely value of p. (Hint: use calculus on the logarithm of the posterior.)

5. Consider the following transition matrix for a 4-state Markov chain:

$$M = \begin{pmatrix} 0.2 & 0.1 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.1 & 0.2 \end{pmatrix}.$$

(a) Is M irreducible?

(b) Find a stationary distribution for M. (Hint: it is unique.)

(c) Is M reversible?

6. Suppose that K is a positive constant and $f(x) = Kx^{-2}$ is a pdf on $1 \le x < \infty$.

(a) Find K.

(b) Find the cumulative distribution function (cdf) of f.

(c) Find the inverse cdf of f.

7. Suppose that X and Y are discrete random variables that have the following joint pdf:

Joint pdf of X and Y

$f(X,Y)$	$Y = 1$	$Y = 2$	$Y = 3$
$X = 1$	0.1	0.1	0.1
$X = 2$	0.3	0.1	0.3

(a) Compute the marginal pdfs f_X and f_Y.

(b) Determine if X and Y are independent.

(c) Compute the complete conditional pdfs $f(X|Y)$ and $f(Y|X)$.