Ma 4111: Advanced Calculus
Homework Assignment 2

Prof. Wickerhauser

Due Tuesday, September 25th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

Put $(a, b) = \{\{a\}, \{a, b\}\}$ for problems 1 and 2.

1. Prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.
2. Define an "ordered n-tuple" $(a_1, a_2, \dots, a_n)$ inductively for $n > 2$ by the formula

$$(a_1, a_2, \dots, a_n) \stackrel{\text{def}}{=} ((a_1, a_2, \dots, a_{n-1}), a_n).$$

   Prove that $(a_1, a_2, \dots, a_n) = (b_1, b_2, \dots, b_n)$ if and only if $a_i = b_i$ for all $i = 1, 2, \dots, n$.

   For problems 3 and 4, define an *equivalence relation* S to be a relation with the following three properties:

   reflexivity: $a \in \operatorname{Dom} S \Rightarrow (a, a) \in S$;

   symmetry: $(a, b) \in S \Rightarrow (b, a) \in S$;

   transitivity: If $(a, b) \in S$ and $(b, c) \in S$ then $(a, c) \in S$.

   Such a relation generalizes "=" and if $(a, b) \in S$ then we say that "a and b are equivalent with respect to S."

3. Determine which of the following plane relations are equivalence relations: (a) $S = \{(x, y) : x^2 = y^2\}$; (b) $S = \{(x, y) : x^2 + y^2 < 1$; (c) $S = \{(x, y) : xy > 0\}$. (d) $S = \{(x, y) : xy < 0\}$.
4. Fix $p \in \mathbf{Z}^+$ and let $S = \{(x, y) \in \mathbf{Z}^+ \times \mathbf{Z}^+ : p|(x - y)\}$. Show that S is an equivalence relation. (If $(x, y) \in S$, then we say that x and y are *congruent modulo* p and write $x \equiv y \pmod{p}$.)

   For problems 5, 6, 7 and 8, let $f : S \to T$ be a function and for each $Y \subset T$ define $f^{-1}(Y) \stackrel{\text{def}}{=} \{x \in S : f(x) \in Y\}$.

5. Prove that $X \subset f^{-1}[f(X)]$ for any $X \subset S$.
6. Prove that $f\left[f^{-1}(Y)\right] \subset Y$ for any $Y \subset T$.
7. Prove that $f\left[f^{-1}(Y)\right] = Y$ for any $Y \subset T$ if and only if $f(S) = T$.

8. Prove that the following five statements are equivalent:
   (a) f is one-to-one on S.
   (b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S.
   (c) $f^{-1}[f(A)] = A$ for every subset A of S.
   (d) If $A \subset S$, $B \subset S$, and $A \cap B = \emptyset$, then $f(A) \cap f(B) = \emptyset$.
   (e) If $A \subset S$, $B \subset S$, and $A \subset B$, then $f(B - A) = f(B) - f(A)$.
9. Suppose that A is countable. Prove that if B is uncountable, then $B - A$ is uncountable.
10. Prove that every uncountable set contains a countably infinite subset.