Ma 4111: Advanced Calculus
Homework Assignment 4

Prof. Wickerhauser

Due Tuesday, October 23rd, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

For Problems 1–2, we say that a subset A of a metric space M is *dense* in a set $S \subset M$ if $A \subset S \subset \overline{A}$, where $\overline{A}$ is the closure of A.

1. Prove that if A is dense in B and B is dense in C, then A is dense in C.
2. A metric space M is called *separable* if there is a countable dense subset of M. Prove that the Lindelöf covering theorem holds in any separable metric space.
3. Suppose that M is a metric space. (a) Prove that if $S \subset M$ is closed and $T \subset M$ is compact, then $S \cap T$ is compact. (b) Prove that if F is an arbitrary collection of closed subsets of M, and some element of F is compact, then $\bigcap_{K \in F} K$ is compact.
4. Suppose that A is an arbitrary subset of a metric space M. Prove (a) that $\partial A = \overline{A} \cap \overline{M - A}$; and (b) that $\partial A = \partial(M - A)$.
5. Prove the following statements about sequences in $\mathbf{C}$:

   (a) $z^n \to 0$ if $|z| < 1$, while $\{z^n\}$ diverges if $|z| > 1$.

   (b) If $z_n \to 0$ and $\{c_n\}$ is bounded, then $c_n z_n \to 0$.
6. Prove that if $x_n \to x$ and $y_n \to y$ are convergent sequences in a metric space (S, d), then $d(x_n, y_n) \to d(x, y)$.
7. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous at least at one point $x_0 \in \mathbf{R}$ and that f satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$. Prove that $f(x) = ax$ for some real number a.
8. Give an example of two metric spaces (S, d_S) and (T, d_T), a continuous function $f : S \to T$, and a Cauchy sequence $\{x_n\} \subset S$ such that $\{f(x_n)\}$ is not a Cauchy sequence in T.
9. Prove that f is continuous on S if and only if the restriction of f to X is continuous on every compact subset $X \subset S$. (Hint: first show that if $x_n \to p \in S$, then the subset $X = \{p, x_1, x_2, \ldots\}$ is compact).
10. Suppose that (S, d_S) and (T, d_T) are metric spaces and $f : S \to T$ is uniformly continuous on S. Prove that if $\{x_n\}$ is a Cauchy sequence in S then $\{f(x_n)\}$ is a Cauchy sequence in T.