Ma 4111: Advanced Calculus
Homework Assignment 5

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Due Tuesday, November 6th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Prove that a metric space S is disconnected if and only if there is a subset $A \subset S$ which is neither empty nor all of S, but which is both open and closed in S.

2. A set $S \subset \mathbf{R}^n$ is called *starlike* if there is some *base point* $x \in S$ such that for every point $y \in S$, the line between x and y is contained in S.

   (a) Prove that every convex subset of $\mathbf{R}^n$ is starlike.

   (b) Prove that every starlike subset of $\mathbf{R}^n$ is connected.

3. Prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and one-to-one on a compact interval $[a, b]$, then f must be strictly monotonic on $[a, b]$.

4. A function $f : \mathbf{R} \to \mathbf{R}$ is said to satisfy a *Lipschitz condition of order* α at a point c in its domain if $(\exists M > 0)(\exists r > 0)(\forall x \in B(c; r), x \neq c)|f(x) - f(c)| < M|x - c|^\alpha$.

   (a) Prove that if f satisfies a Lipschitz condition of order $\alpha > 0$ at c, then f is continuous at c.

   (b) Prove that if f satisfies a Lipschitz condition of order $\alpha > 1$ at c, then f is differentiable at c.

   (c) Find a function f satisfying a Lipschitz condition of order $\alpha = 1$ at c but which is not differentiable at c.

5. Suppose that f is defined on $(0, 1]$ and has a bounded derivative in $(0, 1)$ (*i.e.*, there is a finite $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (0, 1)$). Put $a_n = f(1/n)$ for $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} a_n$ exists. (Hint: use the Cauchy criterion.)

6. Let f be continuous on $[0, 1]$ with $f(0) = 0$ and $f'(x)$ finite at each $x \in (0, 1)$. Suppose $f'(x)$ is an increasing function on $(0, 1)$. Prove that $g(x) \stackrel{\text{def}}{=} f(x)/x$ is an increasing function on $(0, 1)$.

7. Prove that if f has a finite third derivative f''' in $[a, b]$, and $f(a) = f(b) = f'(a) = f'(b) = 0$, then there must be some point $c \in (a, b)$ for which $f'''(c) = 0$.

8. Suppose that the vector-valued function $\mathbf{x}$ is differentiable at each point $t \in (a, b)$, and that $\|\mathbf{x}\|$ is constant on (a, b). Prove that $\mathbf{x}(t) \cdot \mathbf{x}'(t) = 0$ for all $t \in (a, b)$.

9. Define a real-valued function f of two real variables as follows:

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq 0; \qquad f(0, 0) = 0.$$

   (a) Prove that the partial derivatives $D_1 f(x, y)$ and $D_2 f(x, y)$ exist for every $(x, y) \in \mathbf{R}^2$ and find explicit formulas for them.

   (b) Show that f is not continuous at $(0, 0)$.

10. Let S be an open set in $\mathbf{C}$ and let S^* be the set of complex conjugates of points of S: $S^* \stackrel{\text{def}}{=} \{\bar{z} : z \in S\}$. If f is defined on S, define g on S^* by the formula $g(z) = \overline{f(\bar{z})}$. Prove that if f is differentiable at $c \in \text{int}S$, then g is differentiable at $\bar{c}$.