Ma 4111: Advanced Calculus
Homework Assignment 6

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Due Tuesday, November 20th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Determine (with proof) whether the function $f(x) \stackrel{\text{def}}{=} x^{1/5} \cos(\pi/2x)$ if $x \neq 0$, with $f(0) \stackrel{\text{def}}{=} 0$, has bounded variation on the interval $[-1, 1]$.

2. A function $\mu = \mu(x)$ defined on $\mathbf{R}^+$ is called a *Marcinkiewicz multiplier* if there is some $M < \infty$ such that $V_\mu(2^j, 2^{j+1}) < M$ for all integers j; that is, μ has uniformly bounded variation on intervals of the form $[2^j, 2^{j+1}]$.

   (a) Prove that $\mu(x) = \log x$ is a Marcinkiewicz multiplier. Thus such functions do not have to be bounded.

   (b) Prove that μ is a Marcinkiewicz multiplier if and only if there is some $\lambda > 1$ and some $N < \infty$ such that $V_\mu(a, \lambda a) < N$ for every $a > 0$.

3. A real-valued function f defined on $[a, b] \subset \mathbf{R}$ is said to *absolutely continuous* on $[a, b]$ if for every $\epsilon > 0$ there is $\delta > 0$ such that for every finite collection of disjoint open subintervals $(a_i, b_i) \subset [a, b]$ with $\sum_i |b_i - a_i| < \delta$, we have $\sum_i |f(b_i) - f(a_i)| < \epsilon$.

   Prove that a function which is absolutely continuous on $[a, b]$ is continuous and of bounded variation on $[a, b]$.

4. Suppose that $\mathbf{x}$ is a rectifiable path of length L defined on $[a, b]$ and assume that $\mathbf{x}$ is not constant on any subinterval of $[a, b]$. Let $s(x) = \Lambda_{\mathbf{x}}(a, x)$ if $a < x \leq b$ and put $s(a) = 0$. Prove that s^{-1} exists and is continuous on $[0, L]$.

5. Give an example of a bounded function f and an increasing function α defined on $[a, b]$ such that $|f| \in R(\alpha)$ but $f \notin R(\alpha)$.

6. Assume that α has bounded variation on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$. Let $V(x)$ denote the total variation of α on $[a, x]$, where $a < x \leq b$, and put $V(a) = 0$ as usual. Show that

$$\left| \int_a^b f \, d\alpha \right| \leq \int_a^b |f| \, dV \leq MV(b),$$

   where $M = \sup\{|f(t)| : a \leq t \leq b\}$.

7. Let f be a positive continuous function in $[a, b]$. Let M denote the maximum value of f on $[a, b]$. Show that

$$\lim_{n \to \infty} \left(\int_a^b f(x)^n \, dx \right)^{1/n} = M.$$

8. Assume that f has a decreasing derivative which satisfied $f'(x) \geq m > 0$ for all $x \in [a, b]$. Prove that

$$\left| \int_a^b \cos f(x)\, dx \right| \leq \frac{2}{m}.$$

(Hint: Multiply and divide the integrand by $f'(x)$ and use Theorem 7.37(ii)).

9. Prove that the following function is Riemann integrable on $[0, 1]$:

$$f(x) = \begin{cases} 1, & \text{if } x = 0; \\ 0, & \text{if } x \in (0, 1) \text{ is irrational;} \\ 1/n, & \text{if } x \in (0, 1] \text{ is rational, with } x = m/n \text{ in lowest terms.} \end{cases}$$

(Hint: compute the oscillation $\omega_f(x)$ of f at each $x \in [0, 1]$.)

10. Define

$$g(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1, & \text{if } 0 < x \leq 1. \end{cases}$$

(a) Prove that g is Riemann integrable on $[0, 1]$.

(b) Let $f = f(x)$ be as in Problem 9. Prove that $g \circ f$ is not Riemann integrable on $[0, 1]$, despite both $f \in R$ and $g \in R$ on $[0, 1]$.