Ma 4111: Advanced Calculus
Solutions to Homework Assignment 1

Prof. Wickerhauser

Due Tuesday, September 11th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Prove that if $n \in \mathbf{Z}^+$, then

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

Solution: First check case $n = 1$, which is the tautology $a - b = a - b$. For $n > 1$, expand the right-hand side and change variables:

$$(a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k} = \sum_{k=0}^{n-1} a^{k+1} b^{n-1-k} - \sum_{k=0}^{n-1} a^k b^{n-k} = \sum_{k=1}^{n} a^k b^{n-k} - \sum_{k=0}^{n-1} a^k b^{n-k}.$$

Evidently all terms cancel except $k = 0$ and $k = n$, giving the result. □

2. Prove, using induction, that every nonempty subset of $\mathbf{Z}^+$ contains a smallest element. This is called the *well-ordering principle*.

Solution: Prove the equivalent contrapositive statement "If a subset of $\mathbf{Z}^+$ contains no smallest element, then it must be empty."

Suppose that S is a subset of $\mathbf{Z}^+$ with no smallest element. Let $P(n)$ be the statement "n is not an element of S." Then $P(1)$ is true since $1 \in S$ implies S has smallest element 1 as there are no smaller positive integers. Next, suppose that $P(1), \ldots, P(n-1)$ are all true. Then $P(n)$ is true, since otherwise n would be the smallest element of S. Thus $P(n)$ is true for every positive integer n, so S must be empty. □

3. Find the rational number whose decimal expansion is $0.111234\ldots$, where the three digits '234' repeat indefinitely thereafter.

Solution: Let x be the number in question. Note that $10^3 x - x = 111.123$ subtracts away all repetitions of the digit 2, and thus $10^3\left(10^3 - 1\right)x$ is the integer 111,123. Hence,

$$x = \frac{111,123}{10^3\left(10^3-1\right)} = \frac{111,123}{999,000} = \frac{12,347}{111,000}$$

□

4. Prove that $\sqrt{3} - \sqrt{2}$ is irrational.

Solution: The problem is that although we know that $\sqrt{3}$ and $-\sqrt{2}$ are individually irrational, irrationality unlike rationality is not preserved by sums. For example, $\sqrt{2} - 1$ is irrational since if it were rational we could add 1 and falsely conclude that $\sqrt{2}$ is rational. Nonetheless, $(\sqrt{2} - 1) - \sqrt{2}$ is a rational number.

Suppose toward contradiction that $x = \sqrt{3} - \sqrt{2}$ is rational. Then $x^2 = 3 - 2\sqrt{3}\sqrt{2} + 2$ is rational, so we must have that $\sqrt{3}\sqrt{2} = \sqrt{6}$ is rational. But 6 is not the square of an integer, so $\sqrt{6}$ is irrational by theorem 1.10, p.7. □

5. Prove that between any rational number x and irrational number $y > x$ there is both an irrational number $y' \neq y$ and a rational number $x' \neq x$.

Solution: First we show that for every real $z > 0$, there is both a rational and irrational number between 0 and z. But since the set of integers is unbounded, there is an integer $N > 1/z$, and the rational number $1/N$ satisfies $0 < 1/N < z$. For the irrational number, we prove first that the set S of prime numbers is unbounded above. If S were bounded, then we could list all the primes $p_1, p_2, \ldots, p_N$. But then consider the positive integer $q = 1 + p_1 p_2 \cdots p_N$. This is larger than any prime, but it is not divisible by any of the listed primes. Since every positive integer has a unique factorization into primes, this q is either prime and larger than any p_i or it must have a prime factor different from any p_i, $i = 1, 2, \ldots, N$. This is a contradiction, so S must be unbounded above. We are therefore guaranteed that there is a prime integer $p \in S$ such that $p > 1/z^2$. But then $0 < 1/\sqrt{p} < x$, and $1/\sqrt{p}$ is an irrational number by theorem 1.10, p.7.

Finally, put $z = y - x > 0$ and let p, q be the rational and irrational numbers between 0 and z. Then $x' = x + p \neq x$ is rational while $y' = x + q \neq y$ is irrational, and both lie strictly between x and y. □

6. (a) Suppose that A and B are nonempty subsets of $\mathbf{R}^+$ which are bounded above with $a = \sup A$ and $b = \sup B$. For $A \circ B \stackrel{\text{def}}{=} \{x^2 + y^2 : x \in A, y \in B\}$, show that $\sup A \circ B = a^2 + b^2$.

(b) Find two subsets A and B of $\mathbf{R}$ which are bounded above but for which $A \circ B$ is **not** bounded above.

Solution: (a) Since $0 < x \leq a$ and $0 < y \leq b$ for all $x \in A$ and $y \in B$, we conclude that $0 < x^2 + y^2 \leq a^2 + b^2$ by the result stated on p.2 of the textbook, near the bottom. Thus $a^2 + b^2$ is an upper bound for AB. To show that it is the least upper bound, let $\epsilon > 0$ be given. Let $\delta = \frac{\epsilon}{2(a+b)} > 0$ and pick $x \in A$ and $y \in B$ with $0 < a - \delta < x \leq a$ and $0 < b - \delta < y \leq b$. Then $0 < a^2 + b^2 - [2(a+b)\delta - 2\delta^2] < x^2 + y^2 \leq a^2 + b^2$. But $2(a+b)\delta = \epsilon$, so $2(a+b)\delta - 2\delta^2 < \epsilon$, and we have shown that $a^2 + b^2 - \epsilon < x^2 + y^2 < a^2 + b^2$. Since ϵ was arbitrary, we conclude that $a^2 + b^2$ must be the least upper bound.

(b) Let $A = \{-1\}$ and $B = \mathbf{Z}^-$. Both of these sets are bounded above with $\sup A = \sup B = -1$, but the subset $\{(-1)^2 + n^2 : n = -1, -2, -3, \ldots\} \subset A \circ B$ has no upper bound. □

7. Prove the triangle inequality $\|a + b\| \leq \|a\| + \|b\|$ for n-component vectors $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$:

$$\sqrt{\sum_{k=1}^{n} (a_k + b_k)^2} \leq \sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}.$$

Solution: Since all quantities are positive, the triangle inequality holds if its square, the following inequality, holds:

$$\sum_{k=1}^{n} (a_k + b_k)^2 \leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2}$$

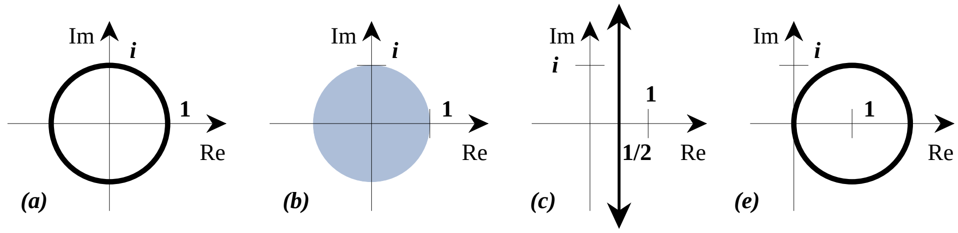

Figure 1: Graphs for Exercise 9.

Expanding the binomial summands in the left-hand sum and then subtracting $\sum_{k=1}^{n} a_k^2$ and $\sum_{k=1}^{n} b_k^2$ from both sides shows that it is enough to prove the following inequality:

$$\sum_{k=1}^{n} 2a_k b_k \leq 2\sqrt{\sum_{k=1}^{n} a_k^2}\sqrt{\sum_{k=1}^{n} b_k^2}$$

But dividing both sides by 2, we see that this is just the Cauchy–Schwarz inequality. □

8. If $z = x + iy$ for real x, y, define the *complex conjugate* of z by $\bar{z} \stackrel{\text{def}}{=} x - iy$. Prove that $\overline{z+w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z}\bar{w}$, $z\bar{z} = |z|^2$, $z + \bar{z} = 2\text{Re}\, z$, and $z - \bar{z} = 2i\text{Im}\, z$.

Solution: Write $w = s + it$, $\bar{w} = s - it$ for $s = \text{Re}\, w$ and $t = \text{Im}\, w$. Then

- $\overline{z+w} = \overline{(x+s) + i(y+t)} = x + s - i(y+t) = x - iy + s - it = \bar{z} + \bar{w}$;
- $\overline{zw} = \overline{(xs - yt) + i(xt + sy)} = (xs - yt) - i(xt + sy) = xs - (-y)(-t) + i(x(-t) + s(-y)) = \bar{z}\bar{w}$;
- $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$;
- $z + \bar{z} = x + iy + x - iy = 2x = 2\text{Re}\, z$;
- $z - \bar{z} = x + iy - (x - iy) = 2iy = 2i\text{Im}\, z$.

□

9. Sketch the following subsets of $\mathbf{C}$: $|z| = 1$, $|z| < 1$, $z + \bar{z} = 1$, $z - \bar{z} = 12$, and $z + \bar{z} = |z|^2$.

Solution: In figure 1, (a) is $\{|z| = 1\}$; (b) is $\{|z| < 1\}$; (c) is $\{z + \bar{z} = 1\}$; (d) would be $\{z - \bar{z} = 12\}$, but this set is empty; and (e) is $\{z + \bar{z} = |z|^2\}$. □

10. Prove that the n n^{th} roots of 1 are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$, where $\alpha = e^{2\pi i/n}$, and that α, α^2 through α^{n-1} each solve the equation $1 + z + \cdots + z^{n-1} = 0$. (Hint: use Problem 1).

Solution: Since $\arg(1) = 0$, $|1| = 1$, and $\sqrt[n]{1} = 1$, theorem 1.52 on p.22 shows that 1 and α^k, $k = 1, 2, \ldots, n-1$ in the stated form are all of the n^{th} roots of 1.

Now notice that $\alpha^k \neq 1$ but $(\alpha^k)^n = 1$ for any $1 \leq k \leq n-1$. By problem 1 with $a = \alpha^k$ and $b = 1$, we see that $0 = (\alpha^k)^n - 1 = a^n - 1 = (a-1)(1 + a + a^2 + \cdots + a^{n-1})$. For this to hold, since $a = \alpha^k \neq 1$, we must have $1 + a + a^2 + \cdots + a^{n-1} = 0$. □