Ma 4111: Advanced Calculus
Solutions to Homework Assignment 2

Prof. Wickerhauser

Due Tuesday, September 25th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

Put $(a, b) = \{\{a\}, \{a, b\}\}$ for problems 1 and 2.

1. Prove that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

   Solution: For $\Leftarrow$, it is evident that $(a, b) = (c, d)$ if $a = c$ and $b = d$. For $\Rightarrow$, we note that $(a, b) = (c, d) \Rightarrow (\{a\} = \{c\}$ or $\{a\} = \{c, d\})$. In the latter case, we must have $a = c = d$, so in either case we have $a = c$. But also, $\{a, b\} = \{c, d\}$ or $\{a, b\} = \{c\}$, so either $b = d$ or else $b = c = a$, and since $\{c, d\} = \{a, b\}$ or $\{c, d\} = \{a\}$ we may then conclude that $d = a = c = b$. □

2. Define an "ordered n-tuple" $(a_1, a_2, \ldots, a_n)$ inductively for $n > 2$ by the formula

$$(a_1, a_2, \ldots, a_n) \stackrel{\text{def}}{=} (\,(a_1, a_2, \ldots, a_{n-1}), a_n).$$

   Prove that $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ for all $i = 1, 2, \ldots, n$.

   Solution: The case $n = 2$ is covered by solution 1 so we consider only $n > 2$. Again, the $\Leftarrow$ direction is evident. For $\Rightarrow$, suppose that $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ and that the result holds for all $(n-1)$-tuples. Then $(a_1, a_2, \ldots, a_{n-1}) = (b_1, b_2, \ldots, b_{n-1})$ so $a_i = b_i$ for $1 \le i \le n-1$ by the inductive hypothesis, while $a_n = b_n$ because the second members of the outer ordered pairs must be equal. □

For problems 3 and 4, define an *equivalence relation* S to be a relation with the following three properties:

reflexivity: $a \in \operatorname{Dom} S \Rightarrow (a, a) \in S$;

symmetry: $(a, b) \in S \Rightarrow (b, a) \in S$;

transitivity: If $(a, b) \in S$ and $(b, c) \in S$ then $(a, c) \in S$.

Such a relation generalizes "=" and if $(a, b) \in S$ then we say that "a and b are equivalent with respect to S."

3. Determine which of the following plane relations are equivalence relations: (a) $S = \{(x, y) : x^2 = y^2\}$; (b) $S = \{(x, y) : x^2 + y^2 < 1$; (c) $S = \{(x, y) : xy > 0\}$. (d) $S = \{(x, y) : xy < 0\}$.

   Solution: (a) Yes, since it is equivalent to $|x| = |y|$, which is transitive, reflexive and symmetric.

   (b) No, since $0.9 \in \operatorname{Dom} S$ but $(0.9, 0.9) \notin S$.

(c) Yes, since it is equivalent to "x and y have the same sign," which is transitive, reflexive, and symmetric.

(d) No, since it is equivalent to "x and y have opposite sign," which is neither transitive, reflexive, nor symmetric. $\square$

4. Fix $p \in \mathbf{Z}^+$ and let $S = \{(x, y) \in \mathbf{Z}^+ \times \mathbf{Z}^+ : p|(x - y)\}$. Show that S is an equivalence relation. (If $(x, y) \in S$, then we say that x and y are *congruent modulo* p and write $x \equiv y \pmod p$.)

   Solution: S is reflexive: $p|(x - y) \iff p|(y - x)$. S is symmetric: $p|(x - x)$ since $x - x = 0 = 0p$. S is transitive: if $p|(x - y)$ and $p|(y - z)$ then $p|\left[(x - y) + (y - z)\right]$ so $p|(x - z)$. $\square$

For problems 5, 6, 7 and 8, let $f : S \to T$ be a function and for each $Y \subset T$ define $f^{-1}(Y) \overset{\text{def}}{=} \{x \in S : f(x) \in Y\}$.

5. Prove that $X \subset f^{-1}\left[f(X)\right]$ for any $X \subset S$.

   Solution: If $x \in X$ then $f(x) \in f(X)$ so $x \in f^{-1}\left[f(X)\right]$ by the definition with $Y = f(X)$. $\square$

6. Prove that $f\left[f^{-1}(Y)\right] \subset Y$ for any $Y \subset T$.

   Solution: By the definition, for every $x \in f^{-1}(Y)$ we have $f(x) \in Y$. Thus $f\left[f^{-1}(Y)\right] \subset Y$. $\square$

7. Prove that $f\left[f^{-1}(Y)\right] = Y$ for any $Y \subset T$ if and only if $f(S) = T$.

   Solution: For $\Rightarrow$, just take $Y = T$. Then $f\left[f^{-1}(T)\right] = T \Rightarrow T \subset \operatorname{Ran} f$. But since $\operatorname{Ran} f \subset T$ we conclude $\operatorname{Ran} f = T$, and since $S = \operatorname{Dom} f$ and $f(\operatorname{Dom} f) = \operatorname{Ran} f$ we know that $f(S) = T$.

   For $\Leftarrow$, suppose that $f\left[f^{-1}(Y)\right] \neq Y$ for some $Y \subset T$. By solution 6, $f\left[f^{-1}(Y)\right]$ must be strictly smaller than Y so there must be some $y \in Y$ with $y \notin f\left[f^{-1}(Y)\right]$. But then there can be no $x \in S$ with $f(x) = y$, since if there were we would have $x \in f^{-1}(Y) \Rightarrow y = f(x) \in f\left[f^{-1}(Y)\right]$. Hence $f(S)$ omits at least y and thus cannot be all of T. $\square$

8. Prove that the following five statements are equivalent:

   (a) f is one-to-one on S.

   (b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S.

   (c) $f^{-1}\left[f(A)\right] = A$ for every subset A of S.

   (d) If $A \subset S$, $B \subset S$, and $A \cap B = \emptyset$, then $f(A) \cap f(B) = \emptyset$.

   (e) If $A \subset S$, $B \subset S$, and $A \subset B$, then $f(B - A) = f(B) - f(A)$.

   Solution: We first remark that $f(X) = \emptyset \iff X = \emptyset$. Now we show in steps that (a) $\Rightarrow$ (b) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$(c)$\Rightarrow$(a). Such a closed loop of implications shows that all the statements are equivalent:

   (a)$\Rightarrow$(b): Suppose $y \in f(A) \cap f(B)$. Then there exists $a \in A$ and $b \in B$ with $f(a) = y = f(b)$. But f is 1-1 so this means $a = b$ and both belong to $A \cap B$. Thus $y = f(a) \in f(A \cap B)$ and we have shown that $f(A) \cap f(B) \subset f(A \cap B)$. At the same time, $A \cap B \subset A$ so $f(A \cap B) \subset f(A)$ and $A \cap B \subset B$ so $f(A \cap B) \subset f(B)$, and therefore $f(A \cap B) \subset f(A) \cap f(B)$.

   (b)$\Rightarrow$(d): If $A \cap B = \emptyset$ then by (b) and the initial remark we compute $f(A) \cap f(B) = f(A \cap B) = f(\emptyset) = \emptyset$.

   (d)$\Rightarrow$(e): Put $C = B - A$, so that $A \cap C = \emptyset$. By (d), we first conclude that $f(A) \cap f(C) = \emptyset$ and thus that $f(C) = f(C) - f(A)$. But $C \subset B \Rightarrow f(C) \subset f(B) \Rightarrow f(C) - f(A) \subset f(B) - f(A)$, so $f(C) \subset f(B) - f(A)$. To see the other inclusion, first notice that $A \subset B \Rightarrow f(A) \subset f(B)$. It also implies that $B = A \cup C$. Thus $f(B) - f(A) = f(A \cup C) - f(A) \subset f(C)$. Together these show that $f(B) - f(A) = f(C) = f(B - A)$.

(e)⇒(c): By solution 5, $A \subset f^{-1}[f(A)]$ for every $A \subset S$, so it is enough to show that $C = f^{-1}[f(A)] - A$ is the empty set for every A. But by (e) and solution 6, $f(C) = f\left(f^{-1}[f(A)] - A\right) = f\left(f^{-1}[f(A)]\right) - f(A) \subset f(A) - f(A) = \emptyset$, so that $C = \emptyset$ by the remark.

(c)⇒(a): If $y = f(x_1) = f(x_2)$, then $x_2 \in f^{-1}(\{y\}) = f^{-1}[f(\{x_1\})] = \{x_1\}$ by (c), so we must have $x_1 = x_2$. □

9. Suppose that A is countable. Prove that if B is uncountable, then $B - A$ is uncountable.

Solution: We prove the contrapositive: if $B - A$ is countable, then B must be countable. But if $B - A$ is countable, then since A is countable the countable (in fact finite) union $(B - A) \cup A$ must also be countable. But then $B \subset (B - A) \cup A$ must be countable, since any subset of a countable set is countable. □

10. Prove that every uncountable set contains a countably infinite subset.

Solution: If S is uncountable it is infinite, and thus nonempy, so choose $a_1 \in S$. Then $S_1 \stackrel{\text{def}}{=} S - \{a_1\}$ is also an infinite set, since otherwise $S \subset S_1 \cup \{a_1\}$ would be finite. We proceed to choose a_{n+1} from $S_n \stackrel{\text{def}}{=} S - \{a_1, \ldots, a_n\}$, which must also be nonempty for every n since S is infinite: if $S_n = \emptyset$ for some n then $S \subset \{a_1, \ldots, a_n\} \cup \emptyset$ must be finite. The sequence $\{a_n : n \in \mathbf{Z}^+\}$ is a countable infinite subset of S. □