Ma 4111: Advanced Calculus
Solutions to Homework Assignment 4

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Due Tuesday, October 23rd, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

For Problems 1–2, we say that a subset A of a metric space M is *dense* in a set $S \subset M$ if $A \subset S \subset \overline{A}$, where $\overline{A}$ is the closure of A.

1. Prove that if A is dense in B and B is dense in C, then A is dense in C.

   Solution: We have $A \subset B \subset \overline{A}$ and $B \subset C \subset \overline{B}$. This implies that $A \subset C$. It thus suffices to show that $\overline{B} \subset \overline{A}$, for then we will have $C \subset \overline{A}$ as well. But if $x \in \overline{B}$, then each ball $B(x;r)$ contains a point y of B and thus a point of $\overline{A}$ since $B \subset \overline{A}$. If $y \in A$ we are done. If $y \in A'$ then we note that $B(x;r)$ is open so there is a ball $B(y;q) \subset B(x;r)$ and this ball $B(y;q)$ contains some point of A. Hence in either case, the arbitrary ball $B(x;r)$ must contain a point of A so we have shown that $x \in \overline{A}$. □

2. A metric space M is called *separable* if there is a countable dense subset of M. Prove that the Lindelöf covering theorem holds in any separable metric space.

   Solution: First of all, denote by $X = \{x_1, x_2, \dots\}$ the dense countable subset of M. Then let F be an open covering of $S \subset M$: $S \subset \bigcup_{A \in F} A$. For each $x \in S$, there is an open set $A_x \in F$ with $x \in A_x$, and since X is dense we have $x_k \in A_x$ for some positive integer k. We can associate to A_x the unique least integer $k = k(x)$ for which $x_k \in A_k$, and this gives us a map $A_x \mapsto \mathbf{Z}^+$ from the subcover $S \subset \bigcup_{x \in S} A_x$ onto a subset of a countable set; hence the subcover must be countable. □

3. Suppose that M is a metric space. (a) Prove that if $S \subset M$ is closed and $T \subset M$ is compact, then $S \cap T$ is compact. (b) Prove that if F is an arbitrary collection of closed subsets of M, and some element of F is compact, then $\bigcap_{K \in F} K$ is compact.

   Solution: (a) Let F be an open cover of $S \cap T$. Now $S \cap T$ is closed since both S and T are closed, so $F \cup \{M - [S \cap T]\}$ is an open covering of S (in fact it covers all of M). This has a finite subcollection $A_1 \cup \ldots \cup A_N \cup (M - [S \cap T])$, $A_i \in F$ which covers S and therefore also covers $S \cap T \subset S$, and since $M - [S \cap T]$ contains no points of $S \cap T$, we conclude that $A_1 \cap \ldots \cap A_N$ is a finite subcollection of F which covers $S \cap T$.

   (b) Let $S \in F$ be one of the compact sets and take $T = \bigcap_{K \in F} K$. An arbitrary intersection of closed sets is closed by theorem 3.35b, so T must be closed in M. Thus $\bigcap_{K \in F} K = S \cap T$ is compact by part (a). □

4. Suppose that A is an arbitrary subset of a metric space M. Prove (a) that $\partial A = \overline{A} \cap \overline{M - A}$; and (b) that $\partial A = \partial(M - A)$.

   Solution: For (a), note that if $x \in \partial A$ then x adheres to A and to $M - A$, since each open ball around x contains a point from each. For (b), note that $M - (M - A) = A$ so that by part (a) we have $\partial(M - A) = \overline{M - A} \cap \overline{M - (M - A)} = \overline{M - A} \cap \overline{A} = \partial(M - A)$. □

5. Prove the following statements about sequences in $\mathbf{C}$:
   (a) $z^n \to 0$ if $|z| < 1$, while $\{z^n\}$ diverges if $|z| > 1$.
   (b) If $z_n \to 0$ and $\{c_n\}$ is bounded, then $c_n z_n \to 0$.

   Solution: (a) We may assume that $|z| > 0$, since otherwise the sequence $\{z^n\}$ is constantly 0. If $|z| < 1$ then for given $\epsilon > 0$, take $M = \frac{\log \epsilon}{\log |z|}$. Then if $n > M$ we compute $|z^n - 0| = |z|^n < |z|^{\frac{\log \epsilon}{\log |z|}} = \epsilon$, since $|z|^{1/\log |z|} = e$. On the other hand, if $|z| > 1$ then $\{z^n\}$ will be unbounded and thus cannot converge: take any $M > 0$ and observe that $|z^n| = |z|^n > |z|^{\frac{\log M}{\log |z|}} = M$ as soon as $n > \frac{\log M}{\log |z|}$.

   (b) Since $\{c_n\}$ is bounded, the sequence $\{|c_n|\}$ is bounded above, so let $c = \sup_n |c_n| \geq 0$. Then $|c_n z_n - 0| = |c_n||z_n| \leq c|z_n|$. Since $z_n \to 0$, for each $\epsilon > 0$ there is some $M > 0$ such that $n > M \Rightarrow |z_n| < \epsilon$. So, given ϵ choose the M for ϵ/c (if $c > 0$) or choose $M = 0$ if $c = 0$. Then we wil be assured that $n > M \Rightarrow c|z_n| < \epsilon$ and thus that $c_n z_n \to 0$. □

6. Prove that if $x_n \to x$ and $y_n \to y$ are convergent sequences in a metric space (S, d), then $d(x_n, y_n) \to d(x, y)$.

   Solution: First note that the triangle inequality gives $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y) \Rightarrow d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$ and also $d(x, y) \leq d(x_n, x) + d(x_n, y_n) + d(y_n, y) \Rightarrow d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y)$. Thus $|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y)$. Now let $\epsilon > 0$ be given. Since $x_n \to x$, we can find an integer $M_x > 0$ such that $n > M_x \Rightarrow d(x_n, x) < \epsilon/2$. Since $y_n \to y$, we can find an integer $M_y > 0$ such that $n > M_y \Rightarrow d(y_n, y) < \epsilon/2$. Put $M = \max\{M_x, M_y\}$. Then $n > M \Rightarrow |d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) < \epsilon/2 + \epsilon/2 = \epsilon$. □

7. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous at least at one point $x_0 \in \mathbf{R}$ and that f satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$. Prove that $f(x) = ax$ for some real number a.

   Solution: We shall first prove that $f(-x) = -f(x)$. But $f(0) = f(0 + 0) = f(0) + f(0) = 2f(0) \Rightarrow f(0) = 0$, and thus $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$.

   We next prove that f is continuous on all of $\mathbf{R}$. Let $x \in \mathbf{R}$ be given along with $\epsilon > 0$. Let $\delta > 0$ satisfy $|y - x_0| < \delta \Rightarrow |f(y) - f(x_0)| < \epsilon$. Then $|z - x| < \delta \Rightarrow |(z - x + x_0) - x_0| < \delta \Rightarrow |f(z - x + x_0) - f(x_0)| = |f(z) - f(x) + f(x_0) - f(x_0)| = |f(z) - f(x)| < \epsilon$.

   Now, if $f(1) = a$, then $f(n) = an$ for every integer n by induction: $f(n + 1) = f(n) + f(1)$. Hence $f(x) = ax$ for each rational $x = n/m$, since

$$an = f(n) = \overbrace{f(n/m) + \ldots + f(n/m)}^{m \text{ times}} = mf(n/m).$$

   Finally, let $x \in \mathbf{R}$ be arbitrary and let $\{x_n\} \subset \mathbf{Q}$ be a sequence of rational points with $x_n \to x$. Then $f(x_n) \to f(x)$ because f is continuous, so $f(x) = \lim_{n\to\infty} ax_n = ax$. □

8. Give an example of two metric spaces (S, d_S) and (T, d_T), a continuous function $f : S \to T$, and a Cauchy sequence $\{x_n\} \subset S$ such that $\{f(x_n)\}$ is not a Cauchy sequence in T.

   Solution: Let $S = (0, 1)$, let $T = \mathbf{R}$, let d_S and d_T be the Euclidean metric and let $f(x) = 1/x$. This f is continuous on S precisely because $0 \notin S$. Then $x_n = 1/n$ defines a Cauchy sequence in S but $f(x_n) = n$ gives an unbounded sequence which cannot therefore be a Cauchy sequence in $\mathbf{R}$. □

9. Prove that f is continuous on S if and only if the restriction of f to X is continuous on every compact subset $X \subset S$. (Hint: first show that if $x_n \to p \in S$, then the subset $X = \{p, x_1, x_2, \ldots\}$ is compact).

   Solution: Following the hint, we let $X = \bigcup_{A\in F} A$ be an open cover of X. Since $p \in X$, we must have $p \in A_0$ for some $A_0 \in F$. The set A_0 is open, so there is some $\epsilon > 0$ such that $B(p; \epsilon) \subset A_0$. Since

$x_n \to p$, there is some $M < \infty$ such that $n > M \Rightarrow x_n \in B(p;\epsilon)$ and thus $\{x_{M+1}, x_{M+2}, \ldots\} \subset A_0$. But we also have open sets $A_1, \ldots, A_M \in F$ with $x_1 \in A_1, \ldots, x_M \in A_M$, so that $X \subset A_0 \cup A_1 \cup \ldots \cup A_M$ is a finite subcover.

For ($\Leftarrow$), note that since f must be continuous on X where $p \in S$ and $\{x_n\}$ is any sequence with $x_n \to p$, we must have $\lim f(x_n) = f(\lim x_n) = f(p)$ for every point $p \in S$. By theorem 4.16 (p.79 in the text), f is therefore continuous on S.

For ($\Rightarrow$), note that if f is continuous on S it is continuous at every point of S and thus is continuous at each point of any subset of S, whether that subset is compact or not. □

10. Suppose that (S, d_S) and (T, d_T) are metric spaces and $f : S \to T$ is uniformly continuous on S. Prove that if $\{x_n\}$ is a Cauchy sequence in S then $\{f(x_n)\}$ is a Cauchy sequence in T.

Solution: Let $\epsilon > 0$ be given and choose $\delta > 0$ such that $d_S(x, y) < \delta \Rightarrow d_T(f(x), f(y)) < \epsilon$. If $\{x_n\} \subset S$ is a Cauchy sequence, then we can find $N < \infty$ such that $n, m > N \Rightarrow d_S(x_n, x_m) < \delta$. But then $n, m > N \Rightarrow d_T(f(x_n), f(x_m)) < \epsilon$. Since ϵ was arbitrary, this proves that $\{f(x_n)\}$ is a Cauchy sequence.

Remark. This result should be compared with the counterexample of Solution 8. □