Ma 4111: Advanced Calculus
Solutions to Homework Assignment 5

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Due Tuesday, November 6th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Prove that a metric space S is disconnected if and only if there is a subset $A \subset S$ which is neither empty nor all of S, but which is both open and closed in S.

   Solution: For ($\Rightarrow$), let $S = A \cup B$ be a disconnection of S. Then A is open. But A is also closed, since $A = (S - B) \cup (A \cap B) = S - B$ because $A \cap B = \emptyset$, and B is open. Also, $A \neq S$ since B is nonempty and disjoint from A.

   For ($\Leftarrow$), suppose A is such a set and put $B = S - A$. Then B is open (since A is closed) and nonempty (since $A \neq S$) and disjoint from A, and we have a disconnection $S = A \cup B$. □

2. A set $S \subset \mathbf{R}^n$ is called *starlike* if there is some *base point* $x \in S$ such that for every point $y \in S$, the line between x and y is contained in S.

   (a) Prove that every convex subset of $\mathbf{R}^n$ is starlike.

   (b) Prove that every starlike subset of $\mathbf{R}^n$ is connected.

   Solution: (a) If S is convex, then for all $x, y \in S$, the line between x and y is contained in S. Fix any $x \in S$; then convex S is starlike with respect to that base point x.

   (b) Any starlike $S \subset \mathbf{R}^n$ is arcwise connected, since if x is the base point, then for any two points $y, z \in S$, the piecewise linear path from y to x to z is contained in S. Hence S is connected by theorem 4.42 of the text. □

3. Prove that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and one-to-one on a compact interval $[a, b]$, then f must be strictly monotonic on $[a, b]$.

   Solution: If $a \geq b$ there is nothing to prove. Otherwise $a < b$ and since f is one-to-one, we must have $f(a) \neq f(b)$. Without loss of generality we can assume that $f(a) < f(b)$ and we will prove that f is strictly increasing. In the other case $f(a) > f(b)$ we can use the same proof on $g = -f$ to show that f is strictly decreasing.

   Suppose toward contradiction that f is not strictly increasing on (a, b). Then there must exist points $x, y \in (a, b)$ with $x < y$ but $f(x) > f(y)$. There are three cases to consider:

   (1) Since f is one-to-one and $a \neq y$, we cannot have $f(a) = f(y)$.

   (2) If $f(a) > f(y)$, then $f(b) > f(a) > f(y)$. Let z be a number between $f(y)$ and $f(a)$. By the intermediate value theorem, the continuous function f must take the same value at two different points, one in (a, y) and one in (y, b). This is impossible, since f is one-to-one.

(3) If $f(a) < f(y)$, then $f(a) < f(x)$ as well since we already know that $f(x) > f(y)$. Since f is continuous, the intermediate value theorem guarantees that f takes each value between $f(x)$ and $f(y)$ in both of the intervals (a, x) and (x, y), again contradicting the hypothesis that f is one-to-one.

All cases are therefore excluded, so we must conclude that f is strictly increasing on (a, b).

It remains to show that f must be strictly increasing on $[a, b]$. We will show that a also belongs to the interval of strict increase. We already know that $f(a) < f(b)$. Suppose toward contradiction that $f(a) \geq f(x)$ for some $x \in (a, b)$. Since f is one-to-one we cannot have $f(a) = f(x)$, so in fact $f(a) > f(x)$. But then since f is continuous, we can find y with $a < y < x$ but so close to x that $|f(y) - f(x)| < [f(a) - f(x)]/2 > 0$. This guarantees that $f(a) > f(y) > f(x)$, which then contradicts the previously established result that f is strictly increasing on (a, b). The endpoint b is treated by a virtually identical argument. □

4. A function $f : \mathbf{R} \to \mathbf{R}$ is said to satisfy a *Lipschitz condition of order* α at a point c in its domain if $(\exists M > 0)(\exists r > 0)(\forall x \in B(c; r), x \neq c)|f(x) - f(c)| < M|x - c|^\alpha$.

   (a) Prove that if f satisfies a Lipschitz condition of order $\alpha > 0$ at c, then f is continuous at c.

   (b) Prove that if f satisfies a Lipschitz condition of order $\alpha > 1$ at c, then f is differentiable at c.

   (c) Find a function f satisfying a Lipschitz condition of order $\alpha = 1$ at c but which is not differentiable at c.

   Solution: (a) Let $\epsilon > 0$ be given and take $\delta = \min\{\left(\frac{\epsilon}{M}\right)^{1/\alpha}, r\}$. Then $\delta > 0$, and $|x - c| < \delta \Rightarrow |f(x) - f(c)| < M|x - c|^\alpha < M\delta^\alpha \leq \epsilon$.

   (b) Let $x \neq c$ be in $B(c; r)$ and divide by $|x - c|$. Then by theorem 1.21 (p.13 of the text) we have

$$-M|x - c|^{(\alpha-1)} < \frac{f(x) - f(c)}{x - c} < M|x - c|^{(\alpha-1)}.$$

   If $\alpha > 1$, then the outer expressions both have 0 as a limit as $x \to c$, so by the sqeeze law of limits the derivative exists at c and equals 0.

   (c) Consider $f(x) = |x|$. This satisfies a Lipschitz condition of order $\alpha = 1$ at $c = 0$, but is not differentiable at 0 since $f'_-(0) = -1 \neq 1 = f'_+(0)$. □

5. Suppose that f is defined on $(0, 1]$ and has a bounded derivative in $(0, 1)$ (*i.e.*, there is a finite $M > 0$ such that $|f'(x)| \leq M$ for all $x \in (0, 1)$). Put $a_n = f(1/n)$ for $n = 1, 2, 3, \ldots$. Prove that $\lim_{n\to\infty} a_n$ exists. (Hint: use the Cauchy criterion.)

   Solution: Say that $|f'(x)| \leq M < \infty$ on $(0, 1)$. For any $n, m > 0$, by the Mean Value Theorem we have $|a_n - a_m| = |f(\frac{1}{n}) - f(\frac{1}{m})| = |(\frac{1}{n} - \frac{1}{m})f'(x)|$ for some x between $\frac{1}{n}$ and $\frac{1}{m}$ (and thus in $(0, 1)$). Thus $|a_n - a_m| \leq M|\frac{1}{n} - \frac{1}{m}|$.

   Now given $\epsilon > 0$, simply choose N so large that $n > N \Rightarrow \frac{1}{n} < \frac{\epsilon}{2M}$. Then $n, m > N \Rightarrow |a_n - a_m| \leq M|\frac{1}{n} - \frac{1}{m}| \leq M|\frac{1}{n}| + M|\frac{1}{m}| < \epsilon$. Thus $\{a_n\}$ is a Cauchy sequence, and thus it has a limit in $\mathbf{R}$. □

6. Let f be continuous on $[0, 1]$ with $f(0) = 0$ and $f'(x)$ finite at each $x \in (0, 1)$. Suppose $f'(x)$ is an increasing function on $(0, 1)$. Prove that $g(x) \stackrel{\text{def}}{=} f(x)/x$ is an increasing function on $(0, 1)$.

   Solution: First of all, $g'(x)$ exists at all $x \in (0, 1)$ because it is the quotient of two differentiable functions and the denominator is never 0 on that set. We will show that $g'(x) \geq 0$ on $(0, 1)$; the result then follows from Theorem 5.7 of the text. But by the quotient rule, $g'(x) = (xf'(x) - f(x))/x^2$. This is nonnegative if $xf'(x) \geq f(x)$, which we will now show. By the Mean Value Theorem, there is some $\xi \in (0, x)$ such that $xf'(\xi) = (x - 0)f'(\xi) = f(x) - f(0) = f(x)$, but since f' increases and $x > 0$ we have $xf'(x) \geq xf'(\xi) = f(x)$. □

7. Prove that if f has a finite third derivative f''' in $[a, b]$, and $f(a) = f(b) = f'(a) = f'(b) = 0$, then there must be some point $c \in (a, b)$ for which $f'''(c) = 0$.

   Solution: By Rolle's theorem, since $f(a) = f(b) = 0$ there must be a point $p \in (a, b)$ for which $f'(p) = 0$. Applying Rolle's theorem to f' on $[a, p]$ and on $[p, b]$, we see that there must be a point $s \in (a, p)$ and a point $t \in (p, b)$ for which $f''(s) = f''(t) = 0$. Applying Rolle's theorem a final time to f'' on $[s, t]$, we see that there must be some $c \in (s, t) \subset (a, b)$ for which $f'''(c) = 0$. □

8. Suppose that the vector-valued function $\mathbf{x}$ is differentiable at each point $t \in (a, b)$, and that $\|\mathbf{x}\|$ is constant on (a, b). Prove that $\mathbf{x}(t) \cdot \mathbf{x}'(t) = 0$ for all $t \in (a, b)$.

   Solution: Since $\|\mathbf{x}(t)\|$ is constant, so is $\|\mathbf{x}(t)\|^2 = \mathbf{x}(t) \cdot \mathbf{x}(t)$. Thus by the product rule (p.114 of the text), for each $t \in (a, b)$ we have $0 = [\mathbf{x}(t) \cdot \mathbf{x}(t)]' = 2\,\mathbf{x}(t) \cdot \mathbf{x}'(t) \Rightarrow \mathbf{x}(t) \cdot \mathbf{x}'(t) = 0$. □

9. Define a real-valued function f of two real variables as follows:

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq 0; \qquad f(0, 0) = 0.$$

   (a) Prove that the partial derivatives $D_1 f(x, y)$ and $D_2 f(x, y)$ exist for every $(x, y) \in \mathbf{R}^2$ and find explicit formulas for them.

   (b) Show that f is not continuous at $(0, 0)$.

   Solution: (a) $D_1 f(x, y)$ and $D_2 f(x, y)$ exist and are continuous away from $(0, 0)$ because there the function f is rational with nonzero denominator in each of the variables x and y. Thus we can use theorem 5.4 of the text to evaluate

$$D_1 f(x, y) = \frac{y^3 - x^2\, y}{\left(x^2 + y^2\right)^2}; \qquad D_2 f(x, y) = \frac{x^3 - x\, y^2}{\left(x^2 + y^2\right)^2},$$

   if $(x, y) \neq (0, 0)$.

   At $(0, 0)$, we must use the definition of the partial derivative. But $f(x, 0) \equiv 0$ and $f(0, y) \equiv 0$, so $D_1 f(0, 0) = D_2 f(0, 0) = 0$.

   (b) f is not continuous at $(0, 0)$, since $f(x, y) \equiv 0$ along the line $\{(x, y) : x = 0\}$, but $f(x, y) \equiv \frac{1}{2}$ along the "punctured" line $\{(x, y) : x = y, x \neq 0\}$. Hence every open neighborhood of $(0, 0)$ contains points where $f = 0$ and points where $f = \frac{1}{2}$, so f cannot be continuous at 0. □

10. Let S be an open set in $\mathbf{C}$ and let S^* be the set of complex conjugates of points of S: $S^* \overset{\text{def}}{=} \{\bar{z} : z \in S\}$. If f is defined on S, define g on S^* by the formula $g(z) = \overline{f(\bar{z})}$. Prove that if f is differentiable at $c \in \text{int}S$, then g is differentiable at $\bar{c}$.

    Solution: First we note that $c \in \text{int}S \Rightarrow \bar{c} \in \text{int}S^*$, since $|\bar{z} - \bar{c}| = |\overline{z - c}| = |z - c|$. Thus the complex conjugates of points in an open ball surrounding c form an open ball surrounding $\bar{c}$.

    Next we observe that $z \in S^* \Rightarrow \bar{z} \in S$, so that

$$\lim_{z \to \bar{c}} \frac{g(z) - g(\bar{c})}{z - \bar{c}} = \lim_{z \to \bar{c}} \frac{\overline{f(\bar{z}) - f(c)}}{\overline{\bar{z} - c}} = \lim_{z \to \bar{c}} \overline{\frac{f(\bar{z}) - f(c)}{\bar{z} - c}} = \overline{f'(c)}.$$

    The first equality follows from the definition of g, the second from the solution to problem 1.29 of the text and by equating the real and imaginary parts of the two limits, while the third follows from the hypothesis that f is differentiable at c. □