Ma 4111: Advanced Calculus
Solutions to Homework Assignment 7

Prof. Wickerhauser

Due Thursday, December 6th, 2012

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Let $a_n = \frac{n!}{n^n}$ for $n > 0$. Prove that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1/e$.

   Solution:
$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \frac{n^n(n+1)!}{(n+1)^{n+1}n!} = \frac{n^n(n+1)n!}{(n+1)(n+1)^n n!} = \left(\frac{n}{n+1}\right)^n.$$
   The last expression equals $\left(1+\frac{1}{n}\right)^{-n}$, which tends to $1/e$ as $n \to \infty$. □

2. Suppose that $\{a_n\}$ is a bounded sequence of real numbers with the property that $\liminf_{n\to\infty} a_n \geq \limsup_{n\to\infty} a_n$. Prove that $\lim_{n\to\infty} a_n$ exists.

   Solution: By theorem 8.3(a) of the text, $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$. From the hypothesis we conclude that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. Since $\{a_n\}$ is bounded both limits are finite, and thus by theorem 8.3(b) we conclude that that $\lim_{n\to\infty} a_n$ exists. □ □

3. Suppose that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Define $\{b_n\}$ by
$$b_n = \begin{cases} a_n^2, & \text{if } n \text{ is even,} \\ -a_n^n, & \text{if } n \text{ is odd.} \end{cases}$$
   Prove that $\sum_{n=1}^{\infty} b_n$ converges absolutely.

   Solution: Since $\sum |a_n|$ converges, we must have $\lim_{n\to\infty} |a_n| = 0$. Thus we can find an $N < \infty$, without loss of generality assuming $N > 2$, such that $n > N \Rightarrow |a_n| < 1 \Rightarrow |a_n^2| < |a_n|$ and $n > N \Rightarrow |a_n| < 1 \Rightarrow |-a_n^n| < |a_n|$. Thus for all $n > N$, $0 \leq |b_n| \leq |a_n|$, and so the series $\sum |b_n|$ converges by theorem 8.20. □

4. Determine with proof whether the following series converges:
$$\sum_{n=1}^{\infty} \left(\sqrt{1+n^6} - n^3\right)$$

   Solution: Rewrite
$$\sum_{n=1}^{\infty} \left(\sqrt{1+n^6} - n^3\right) = \sum_{n=1}^{\infty} \frac{\left(\sqrt{1+n^6} - n^3\right)\left(\sqrt{1+n^6} + n^3\right)}{\sqrt{1+n^6} + n^3} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^6} + n^3}.$$
   This converges by comparison with $\sum \frac{1}{n^3}$. □

5. Determine with proof whether the following series converges:

$$\sum_{n=1}^{\infty}\left(\sqrt{1+n}-\sqrt{n}\right)$$

Solution: Rewrite

$$\sum_{n=1}^{\infty}\left(\sqrt{1+n}-\sqrt{n}\right)=\sum_{n=1}^{\infty}\frac{\left(\sqrt{1+n}-\sqrt{n}\right)\left(\sqrt{1+n}+\sqrt{n}\right)}{\sqrt{1+n}+\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{\sqrt{1+n}+\sqrt{n}}.$$

This diverges by comparison with $\sum \frac{1}{\sqrt{n}}$; the summands are larger than $\frac{1}{2n}$ for all $n>0$. Alternatively, notice that the series telescopes, so that for any $M \geq 1$,

$$\sum_{n=1}^{M}\left(\sqrt{1+n}-\sqrt{n}\right)=\sqrt{M+1}-\sqrt{1}.$$

This partial sum increases without bound as $M \to \infty$, so the series diverges. □

6. Determine with proof whether the following series converges:

$$\sum_{n=2}^{\infty}(\log n)^{-\log n}$$

Solution: We use the integral test, with the substitution $x \leftarrow e^y$, $dx = e^y\,dy$:

$$\int_e^{\infty}(\log x)^{-\log x}\,dx=\int_1^{\infty}y^{-y}e^y\,dy=\int_1^{\infty}e^{y(1-\log y)}\,dy.$$

This converges by comparison with $\int_1^{\infty}e^{-y}\,dy<\infty$, since $1-\log y<-1$ for all $y>e^2$. □

7. Find a double sequence $\{a_{n,m}\}$ such that $\lim_{n\to\infty}a_{n,m}=0$ for all fixed m and $\lim_{m\to\infty}a_{n,m}=0$ for all fixed n, but $\lim_{n,m\to\infty}a_{n,m}$ does not exist.

Solution: Consider $a_{n,m}=\frac{1}{n-m+0.5}$. For each fixed n, $\frac{1}{n-m+0.5}\to 0$ as $m\to\infty$. Likewise, for each fixed m, $\frac{1}{n-m+0.5}\to 0$ as $n\to\infty$. Hence 0 is the only candidate for the limit of the double sequence. However, $a_{n,n}=2$ no matter how large a value we take for n. □

8. Find the Cesàro sum of the complex-valued series $\sum_{n=0}^{\infty}i^n$, where $i^2=-1$.

Solution: The partial sums $s_m=\sum_{n=0}^{m}i^n$ take the values 1, $1+i$, i, and 0, depending on whether the remainder left after dividing m by 4 is 0,1,2, or 3, respectively. Therefore, for any integer written $m=4k+m'\geq 0$ with $0\leq m'<4$, we have

$$\sigma_m=\frac{s_0+s_1+\ldots+s_m}{m}=\frac{k(1+[1+i]+i+0)}{4k+m'}+\frac{s_{m'}}{4k+m'}.$$

But since $s_{m'}/m\to 0$ as $m\to\infty$, and $k/(4k+m')\to\frac{1}{4}$ as $m\to\infty$, the Cesàro sum is evidently $(1+i)/2$. □

9. Prove that $\prod_{n=2}^{\infty}(1-n^{-2})$ converges and evaluate it.

Solution: Convergence follows from theorem 8.52, since the series $\sum n^{-2}$ converges. Rewriting the partial product yields $\prod_{n=2}^{K}\frac{n^2-1}{n^2}=\prod_{n=2}^{K}\frac{(n-1)(n+1)}{n^2}$, wherein one factor of the numerator cancels part of a past denominator, while the other cancels part of a future denominator. Hence the product telescopes down to $\frac{(2-1)}{2}\frac{(K+1)}{K}$, which tends to the limit $\frac{1}{2}$ as $K\to\infty$. □

10. Prove that if a double series converges absolutely, then it converges.

Solution: Write $\sum_{n,m} f(n,m)$ for the double series and $\{s(p,q)\}$ for the double sequence of partial sums. We will show that $s(p,q)$ satisfies the Cauchy condition, namely, that for any $\epsilon > 0$ we can find $N < \infty$ such that $|s(p_1,q_1) - s(p_2,q_2)| < \epsilon$ whenever $p_1 > p_2 \geq N$ and $q_1 > q_2 \geq N$. But this follows from the Cauchy condition which is satisfied by $S(p,q)$, the sequence of partial sums of the series $\sum |f(n,m)|$, and from the triangle inequality:

$$\begin{aligned}
|s(p_2,q_2) - s(p_1,q_1)| &= \left| \sum_{p_1+1}^{p_2} \sum_{1}^{q_2} f(p,q) + \sum_{1}^{p_1} \sum_{q_1+1}^{q_2} f(p,q) \right| \\
&\leq \sum_{p_1+1}^{p_2} \sum_{1}^{q_2} |f(p,q)| + \sum_{1}^{p_1} \sum_{q_1+1}^{q_2} |f(p,q)| \\
&= |S(p_2,q_2) - S(p_1,q_1)| \,.
\end{aligned}$$

Hence $\{s(p,q)\}$ is a convergent double sequence, and so $\sum f(n,m)$ is a convergent double series. $\square$