Ma 4121: Introduction to Lebesgue Integration
Homework Assignment 1

Prof. Wickerhauser

Due Thursday, January 31st, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Let S be a countable set of real numbers. Prove that S has measure zero.

2. Suppose that subset S of $\mathbf{R}$ is a set of measure zero.

   (a) Prove that every subset of S also has measure zero.

   (b) For fixed $x \in \mathbf{R}$, define $S + x = \{s + x : s \in S\}$. Prove that $S + x$ has measure zero.

   (c) For fixed finite $p > 0$, define $pS = \{ps : s \in S\}$. Prove that pS has measure zero.

3. Suppose that $f_n \to f$ and $g_n \to g$ both converge uniformly on S. Prove that $f_n - g_n$ converges uniformly to $f - g$ on S.

4. Define $h_n(x) = f_n(x)g_n(x)$, where $f_n(x) = (1 + \frac{1}{n})x$ and

$$g_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational,} \\ b + \frac{1}{n}, & \text{if } x = \frac{a}{b} \text{ in lowest terms, with } b > 0 \text{ and } a \neq 0. \end{cases}$$

   (a) Prove that $\{f_n\}$ and $\{g_n\}$ converge uniformly on every bounded interval.

   (b) Prove that $\{h_n\}$ does **not** converge uniformly on any bounded interval.

5. Prove:

   (a) If $\{f_n\}$ and $\{g_n\}$ are both uniformly bounded sequences of functions on S, and if $f_n \to f$ and $g_n \to g$ both converge uniformly on S, then $f_n g_n \to fg$ will converge uniformly on S.

   (b) If $\{f_n\}$ and $\{1/g_n\}$ are both uniformly bounded sequences of functions on S, and if $f_n \to f$ and $g_n \to g$ both converge uniformly on S, then $f_n/g_n \to f/g$ will converge uniformly on S.

6. Assume that

   i. $f_n \to f$ uniformly on S;

   ii. Each f_n is continuous on S;

   iii. The sequence $\{x_n\} \subset S$ converges to $a \in S$.

   Prove that $f_n(x_n) \to f(a)$ as $n \to \infty$.

7. Find counterexamples to show that each assumption i, ii, and iii is necessary in the previous problem.

8. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, with $f(0) \stackrel{\text{def}}{=} 0$. Prove that $f^{(n)}(0)$ exists and equals 0 for each $n = 0, 1, 2, \ldots$.

9. Suppose $f(x)$ is represented by the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 = a_1 = 3$ while $a_n = 1$ for $n \geq 2$. What is the power series for $1/f(x)$, and its radius of convergence?

10. Let $f_n(x)$ be the real-valued function defined on $[0, 1]$ by the formula

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 2^{-n}; \\ 2^{n/2}, & \text{if } 2^{-n} \leq x \leq 2^{1-n}; \\ 0, & \text{if } 2^{1-n} < x \leq 1, \end{cases}$$

for $n = 1, 2, \ldots$. Prove that $\{f_n\}$ converges pointwise to 0 on $[0, 1]$, but $\text{l.i.m.}_{n\to\infty} f_n \neq 0$.