Ma 4121: Introduction to Lebesgue Integration
Homework Assignment 2

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Due Thursday, February 14th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Suppose that $\{f_n\}$ and $\{g_n\}$ are increasing sequences of functions on an interval I, and put $u_n = \max(f_n, g_n)$. Prove that if $f_n \nearrow f$ *a.e.* on I, and $g_n \nearrow g$ *a.e.* on I, then $u_n \nearrow \max(f, g)$ *a.e.* on I.

2. Let $I = [0, 1]$. Find a function $f \in U(I)$ such that $-f \notin U(I)$.

3. Suppose that $\{f_n\} \subset L(I)$ satisfies $(\forall n) f_n(x) \geq 0$ *a.e.* on I, and $f_n \to f$ *a.e.* on I, and $(\exists A < \infty)(\forall n) \int_I f_n \leq A$. Prove that the limit function f belongs to $L(I)$ and that $\int_I f \leq A$. (this is called *Fatou's Lemma*).

4. Find, with proof, all $p \in \mathbf{R}$ for which the Lebesgue integral $\int_0^\infty x^p \sin(x^2)\, dx$ exists.

5. Prove that the following Lebesgue integrals exist:
$$\int_0^1 (x \log x)^2\, dx, \qquad \int_0^1 \log x \log(1-x)^2\, dx, \qquad \int_0^1 \frac{\sqrt{1-x}}{\log x}\, dx.$$

6. For each of the Lebesgue integrals and intervals I below, determine with proof the set S of values $s \in \mathbf{R}$ for which it must exist for every function $f \in L(I)$. For each s not in S, find a bounded continuous f for which the Lebesgue integral fails to exist.
$$\int_0^1 f(x) \cos(2\pi s x)\, dx, \qquad \int_0^\infty f(x) e^{sx}\, dx, \qquad \int_0^\infty \frac{f(x)}{x^2 + s^2}\, dx.$$

7. Suppose that f is continuous on $[0, 1]$, $f(0) = 0$, and $f'(0)$ exists. Prove that the Lebesgue integral $\int_0^1 f(x) x^{-3/2}\, dx$ exists.

8. Suppose that $f \in L([0, 1])$ and put $g_n(x) \stackrel{\text{def}}{=} f(x) \sin(nx)$ for integers n.

   (a) Prove that g_n also belongs to $L([0, 1])$.

   (b) Prove that $\int_0^1 g_n \to 0$ as $|n| \to \infty$. (Hint: first prove the result for step functions.)

9. Suppose that $I = \mathbf{R}$ and $f \in L(I)$. Put $f_y(x) \stackrel{\text{def}}{=} f(x - y)$ for $y \in \mathbf{R}$. Prove that $\int_I |f_y - f| \to 0$ as $y \to 0$. (Hint hint: see previous hint.)

10. If f is Lebesgue-integrable on an open interval I and if f' exists *a.e.* on I, prove that f' is measurable on I.