Ma 4121: Introduction to Lebesgue Integration
Homework Assignment 3

Prof. Wickerhauser

Due Thursday, February 28th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Let $\sim$ denote the relation on $\mathbf{R}$ defined by

$$x \sim y \iff x - y \in \mathbf{Q}$$

   Prove that $\sim$ is an equivalence relation, namely:

   (i) $(\forall x \in \mathbf{R}) x \sim x$

   (ii) $(\forall x, y \in \mathbf{R}) x \sim y \iff y \sim x$

   (iii) $(\forall x, y, z \in \mathbf{R})(x \sim y \text{ and } y \sim z) \Rightarrow x \sim z$

2. Given $x \in \mathbf{R}$, define the equivalence class $[x] = \{y \in \mathbf{R} : x \sim y\}$, where $\sim$ is the equivalence relation of exercise 1 above.

   (a) Prove that $[x]$ is countably infinite for every $x \in \mathbf{R}$.

   (b) Prove that the number of distinct equivalent classes is uncountable.

3. Suppose that $\{f_n\} \subset L(I)$ satisfies $(\forall n) f_n(x) \geq 0$ *a.e.* on I, and $f_n \to f$ *a.e.* on I, and $(\exists A < \infty)(\forall n) \int_I f_n \leq A$. Prove that the limit function f belongs to $L(I)$ and that $\int_I f \leq A$. (this is called *Fatou's Lemma*).

4. Find, with proof, all $p \in \mathbf{R}$ for which the Lebesgue integral $\int_0^\infty x^p \sin(x^2)\, dx$ exists.

5. Prove that the following Lebesgue integrals exist:

$$\int_0^1 (x \log x)^2 \, dx, \qquad \int_0^1 \log x \log(1-x)^2 \, dx, \qquad \int_0^1 \frac{\sqrt{1-x}}{\log x} \, dx.$$

6. For each of the Lebesgue integrals and intervals I below, determine with proof the set S of values $s \in \mathbf{R}$ for which it must exist for every function $f \in L(I)$. For each s not in S, find a bounded continuous f for which the Lebesgue integral fails to exist.

$$\int_0^1 f(x) \cos(2\pi s x) \, dx, \qquad \int_0^\infty f(x) e^{sx} \, dx, \qquad \int_0^\infty \frac{f(x)}{x^2 + s^2} \, dx.$$

7. Suppose that f is continuous on $[0, 1]$, $f(0) = 0$, and $f'(0)$ exists. Prove that the Lebesgue integral $\int_0^1 f(x) x^{-3/2} \, dx$ exists.

8. Suppose that $f \in L([0,1])$ and put $g_n(x) \stackrel{\text{def}}{=} f(x)\sin(nx)$ for integers n.

   (a) Prove that g_n also belongs to $L([0,1])$.

   (b) Prove that $\int_0^1 g_n \to 0$ as $|n| \to \infty$. (Hint: first prove the result for step functions.)

9. Put $I = [0,1]$. Suppose that f is continuous on I with $f(0) = 0$, and that $f'(0)$ exists and is finite. (Here we mean the one-sided derivative

$$f'(0) \stackrel{\text{def}}{=} \lim_{h \to 0+} \frac{f(h) - f(0)}{h}.)$$

   Prove that $g(x) \stackrel{\text{def}}{=} f(x)/x^{3/2}$ belongs to $L(I)$.

10. If f is Lebesgue-integrable on an open interval I and if f' exists *a.e.* on I, prove that f' is measurable on I.