Ma 4121: Introduction to Lebesgue Integration
Homework Assignment 4

Prof. Wickerhauser

Due Thursday, March 28th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Find a counterexample to show that the union of two σ-algebras for a set X need not be a σ-algebra on X. (Hint: it is enough to consider a three-point set X.)

2. Show that the Borel sets $\mathcal{B}(\mathbf{R})$ are generated by the compact subsets of $\mathbf{R}$.

3. Let $(X, \mathcal{A})$ be a measurable space and suppose $\mu : \mathcal{A} \to [0, +\infty]$ is a countably additive function on the σ-algebra $\mathcal{A}$.

   (a) Show that if μ satisfies $\mu(A) < \infty$ for some $A \in \mathcal{A}$, then $\mu(\emptyset) = 0$. (This implies that μ is a measure.)

   (b) Find an example μ for which $\mu(\emptyset) \neq 0$. (Thus the first property of a measure does not follow from countable additivity and non-negativity.)

4. Let $(X, \mathcal{A})$ be a measurable space. Say that a sequence of measures $\{\mu_n : n = 1, 2, \ldots\}$ is *increasing* iff

   $$(\forall A \in \mathcal{A})(\forall n)\ \mu_{n+1}(A) \geq \mu_n(A)$$

   (a) Show that if $\{\mu_n\}$ is an increasing sequence of measures, then

   $$\mu(A) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu_n(A)$$

   defines a measure on $(X, \mathcal{A})$.

   (b) Let $\{\mu_n : n = 1, 2, \ldots\}$ be any sequence of measures on $(X, \mathcal{A})$. Prove that

   $$\mu(A) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \mu_n(A)$$

   defines a measure on $(X, \mathcal{A})$.

5. Let $(X, \mathcal{A})$ be a measurable space and define the function $\delta_x : \mathcal{A} \to [0, +\infty]$ by

   $$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

   Prove that δ_x is a measure.

6. Let $\{x_n : n = 1, 2, \ldots\} \subset \mathbf{R}$ be a sequence of points and define a measure μ on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ by

$$\mu = \sum_n \delta_{x_n},$$

where δ_x is the measure defined in exercise 5.

(a) Show that μ assigns finite values to bounded intervals if and only if $|x_n| \to +\infty$ as $n \to \infty$.

(b) For what sequences is this μ a σ-finite measure?

7. Show that a subset $B \subset \mathbf{R}$ is Lebesgue measurable if and only if

$$\lambda^*(I) = \lambda^*(I \cap B) + \lambda^*(I \cap B^c)$$

for every open interval $I \subset \mathbf{R}$.

8. Let $(X, \mathcal{A}, \mu)$ be a measure space and let μ^* denote the outer measure defined by μ using $\mathcal{A}$, namely

$$\mu^*(S) = \inf\{\mu(A) : S \subset A, A \in \mathcal{A}\}$$

Define the *inner measure* $\mu_* : 2^X \to [0, +\infty]$ by

$$\mu_*(S) = \sup\{\mu(A) : A \subset S, A \in \mathcal{A}\}$$

(a) Prove that $\mu_*(S) \leq \mu^*(S)$ for every $S \subset X$.

(b) Prove that for any subset $S \subset X$ (not necessarily in $\mathcal{A}$) there are sets $A_0, A_1 \in \mathcal{A}$ satisfying $A_0 \subset S \subset A_1$ and

$$\mu(A_0) = \mu_*(S); \qquad \mu^*(S) = \mu(A_1).$$

9. Say that the measure space $(X, \mathcal{A}, \mu)$ is *complete* iff whenever $S \subset X$ and there is some $A \in \mathcal{A}$ with $S \subset A$ and $\mu(A) = 0$, we may conclude that $S \in \mathcal{A}$.

(a) Prove that if μ^* is an outer measure on X, and $\mathcal{A}$ is the σ-algebra of μ^*-measurable sets, then $(X, \mathcal{A}, \mu^*)$ is complete.

(b) Decide whether $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$ is complete.

10. Define the *completion* of the σ-algebra $\mathcal{A}$ under μ to be the collection $\mathcal{A}_\mu$ of subsets $S \subset X$ for which there exist $E, F \in \mathcal{A}$ satisfying $E \subset S \subset F$ and $\mu(F \setminus E) = 0$.

(a) Prove that $\mathcal{A} \subset \mathcal{A}_\mu$.

(b) Prove that $S \in \mathcal{A}_\mu$ if and only if $\mu_*(S) = \mu^*(S)$. (Hint: assume exercise 8.)

(c) Prove that $\mathcal{A}_\mu$ is a σ-algebra.