Ma 4121: Introduction to Lebesgue Integration
Homework Assignment 5

Prof. Wickerhauser

Due Thursday, April 11th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Verify that the trigonometric system $\{\frac{1}{\sqrt{\pi}}\sin nx : n = 1, 2, \ldots\} \cup \{\frac{1}{\sqrt{\pi}}\cos nx : n = 1, 2, \ldots\} \cup \{\frac{1}{\sqrt{2\pi}}\}$ is orthonormal in $L^2([0, 2\pi])$.

2. Let $\{\phi_0, \phi_1, \ldots\} \subset X$ be an orthonormal system of functions, where X is an inner product space in which $\|f\| = 0 \iff f = 0$. Prove that the following three statements are equivalent:
   (a) If $\langle f, \phi_n\rangle = \langle g, \phi_n\rangle$ for all $n = 0, 1, 2, \ldots$, then $f = g$.
   (b) If $\langle f, \phi_n\rangle = 0$ for all n, then $f = 0$.
   (c) If T is an orthonormal system such that $\{\phi_0, \phi_1, \ldots\} \subset T$, then $T = \{\phi_0, \phi_1, \ldots\}$.

   For Problems 3 and 4, define $C([0, 1])$ to be the space of complex-valued continuous functions on the compact interval $[0, 1]$, with $\langle f, g\rangle \stackrel{\text{def}}{=} \int_0^1 f(t)\bar{g}(t)\,dt$ for $f, g \in C([0, 1])$, and $\|f\| \stackrel{\text{def}}{=} \sqrt{\langle f, f\rangle}$.

3. For $f \in C([0, 1])$, prove that $\|f\| = 0 \iff f = 0$.

4. Prove that the set $\{e^{2\pi int} : n \in \mathbf{Z}\} \subset C([0, 1])$ is an orthonormal system satisfying all three conditions of Problem 2.

5. Put $I = [0, 1] \subset \mathbf{R}$. Suppose that $\mathcal{H} = \{\psi_n : n = 0, 1, 2, \ldots\} \subset L^2(I)$ is the orthogonal system of Haar functions defined for $n = 0$ by $\psi_0 = \mathbf{1}_I$, and for $0 < n = 2^j + k$ with $0 \le k < 2^j$ is defined by
$$\psi_n(x) = \begin{cases} 1, & \text{if } \frac{k}{2^j} \le x < \frac{k+\frac{1}{2}}{2^j}, \\ -1, & \text{if } \frac{k+\frac{1}{2}}{2^j} \le x < \frac{k+1}{2^j}, \\ 0, & \text{otherwise.} \end{cases}$$
   Show that if $f \in L^2(I)$ and $\langle f, \psi_n\rangle = 0$ for all $n = 0, 1, 2, \ldots$, then $f = 0$ *a.e.* on I.

6. Show that $x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, if $0 < x < 2\pi$.

7. Show that $\dfrac{x^2}{2} = \pi x + 2\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^2} - 2\sum_{n=1}^{\infty} \dfrac{1}{n^2}$, if $0 \le x \le 2\pi$. Conclude that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$. (Hint: integrate the formula from Problem 6.)

For Problems 8 and 9, define a 2π-periodic function f as follows:

$$f(t) = \begin{cases} 1, & \text{if } 0 < t < \pi; \\ -1, & \text{if } -\pi < t < 0; \\ 0, & \text{if } t = -\pi,\ t = 0, \text{ or } t = \pi. \end{cases}$$

8. Show that $f(x) = \dfrac{4}{\pi} \sum_{n=1}^{\infty} \dfrac{\sin(2n-1)x}{2n-1}$ for every $x \in \mathbf{R}$.

9. Let $s_n(x)$ be the partial sum of the first n terms of the Fourier series of the function f defined above. Show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \left[\max_{|x|<\epsilon} s_n(x) - \min_{|x|<\epsilon} s_n(x) \right] = \frac{4}{\pi} \int_0^{\pi} \frac{\sin t}{t}\, dt.$$

(Hint: see problem 11.19 on pp.338–339 of the text.)

This result is known as *Gibbs' phenomenon.*

10. Prove that if $f \in L([0, 2\pi])$ and $f'(x_0)$ exists at some point $x_0 \in (0, 2\pi)$, then the Fourier series generated by f converges at x_0.