Ma 4121: Introduction to Lebesgue Integration
Homework Assignment 6

Prof. Wickerhauser

Due Thursday, April 25th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. For fixed $c \in (0,1)$, define $f : \mathbf{R}^2 \to \mathbf{R}$ as follows:

$$f(x,y) \stackrel{\text{def}}{=} \begin{cases} (1-y)^c/(x-y)^c, & \text{if } 0 < x < 1 \text{ and } 0 \le y < x; \\ 0, & \text{otherwise.} \end{cases}$$

   Prove that $f \in L(\mathbf{R}^2)$ and evaluate $\int_{\mathbf{R}^2} f$.

2. Suppose that $S \subset \mathbf{R}^2$ is a measurable set with the property that $\lambda(S_y) = 0$ for almost every $y \in \mathbf{R}$, where λ is 1-dimensional Lebesgue measure on $\mathbf{R}$, and

$$S_y \stackrel{\text{def}}{=} \{x \in \mathbf{R} : (x,y) \in S\}.$$

   Prove that the 2-dimensional Lebesgue measure of S is zero. (Note: This is a partial converse to Theorem 15.5 on p.412 of our text.)

3. Suppose that $f_i : \mathbf{R} \to \mathbf{R}$ is defined and bounded on the compact interval $[a_i, b_i] \subset \mathbf{R}$. If $f_i \in L([a_i, b_i])$ for $i = 1, \ldots, n$, prove that

$$\int_Q f_1(x_1) \cdots f_n(x_n)\, d(x_1, \ldots, x_n) = \left(\int_{a_1}^{b_1} f_1(x_1)\, dx_1 \right) \cdots \left(\int_{a_n}^{b_n} f_n(x_n)\, dx_n \right),$$

   where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbf{R}^n$.

4. (a) Prove that $\int_{\mathbf{R}^2} e^{-x^2-y^2} = \pi$ by transforming the integral to polar coordinates

   (b) Use part(a) to prove that $\int_{\mathbf{R}} e^{-x^2} = \sqrt{\pi}$.

   (c) Use part (b) to prove that $\int_{\mathbf{R}^n} e^{-\|x\|^2} = \pi^{n/2}$.

   (d) Evaluate $\int_{\mathbf{R}} e^{-tx^2}$ for $t > 0$, and find t for which the value is 1.

5. Let $V_n(a)$ denote the volume of the ball of radius a in $\mathbf{R}^n$, that is, the n-dimensional Lebesgue measure of the open set $\{x \in \mathbf{R}^n : \|x\| < a\}$.

   (a) Prove that $V_n(a) = a^n V_n(1)$.

   (b) Prove that, for $n \ge 3$, we have the formula

$$V_n(1) \;=\; V_{n-2}(1) \times \int_0^{2\pi} \left[\int_0^1 (1-r^2)^{n/2-1} r\, dr \right] d\theta \;=\; V_{n-2}(1) \frac{2\pi}{n}.$$

(c) Use the recursion in part (b) to conclude that

$$V_n(1) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n + 1)},$$

where Γ is the special function defined on p.277 of our text.

6. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$f(x, y) = \begin{cases} e^y \sin x, & \text{if } x \text{ is rational;} \\ e^{-x^2 - y^2}, & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $f \in L(\mathbf{R}^2)$ and compute $\int_{\mathbf{R}^2} f$.

7. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ for $0 \le x \le 1$, $0 < y \le 1$, and put $f(0, 0) = 0$. Prove that both iterated integrals

$$\int_{y=0}^{1} \left[\int_{x=0}^{1} f(x, y)\, dx \right] dy, \quad \text{and} \quad \int_{x=0}^{1} \left[\int_{y=0}^{1} f(x, y)\, dy \right] dx$$

exist but are not equal. Conclude that $f \notin L([0, 1] \times [0, 1])$.

8. Let $f(x, y) = e^{-xy} \sin x \sin y$ for $x \ge 0$ and $y \ge 0$, and let $f(x, y) = 0$ otherwise. Prove that both iterated integrals

$$\int_{y \in \mathbf{R}} \left[\int_{x \in \mathbf{R}} f(x, y)\, dx \right] dy, \quad \text{and} \quad \int_{x \in \mathbf{R}} \left[\int_{y \in \mathbf{R}} f(x, y)\, dy \right] dx$$

exist and are equal, but that $f \notin L(\mathbf{R}^2)$. Explain why this does not contradict the Tonelli-Hobson test (theorem 15.8, p.415).

9. Let $I = [0, 1] \times [0, 1]$, let $f(x, y) = (x - y)/(x + y)^3$ if $(x, y) \in I \setminus \{(0, 0)\}$, and let $f(0, 0) = 0$. Prove that $f \notin L(I)$ by considering the integrals

$$\int_{y=0}^{1} \left[\int_{x=0}^{1} f(x, y)\, dx \right] dy, \quad \text{and} \quad \int_{x=0}^{1} \left[\int_{y=0}^{1} f(x, y)\, dy \right] dx$$

10. Let $I = [0, 1] \times [1, +\infty)$ and let $f(x, y) = e^{-xy} - 2e^{-2xy}$ if $(x, y) \in I$. Prove that $f \notin L(I)$ by considering the integrals

$$\int_{y=1}^{\infty} \left[\int_{x=0}^{1} f(x, y)\, dx \right] dy, \quad \text{and} \quad \int_{x=0}^{1} \left[\int_{y=1}^{\infty} f(x, y)\, dy \right] dx$$