Ma 4121: Introduction to Lebesgue Integration
Solutions to Homework Assignment 1

Prof. Wickerhauser

Due Thursday, January 31st, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Let S be a countable set of real numbers. Prove that S has measure zero.

   Solution: Let $\epsilon > 0$ be given, then put $\delta = \epsilon/3$.

   Since S is countable, we may list its elements $S = \{x_1, x_2, \ldots\}$.

   For $k = 1, 2, \ldots$, let $I_k = (x_k - \delta/2^k, x_k + \delta/2^k)$ and note that $|I_k| = 2\delta/2^k$.

   Since $x_k \in I_k$ for all $k = 1, 2, \ldots$, we have

   $$S \subset \bigcup_{k=1}^{\infty} I_k.$$

   But also,

   $$\sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} 2\delta/2^k = 2\delta = \frac{2}{3}\epsilon < \epsilon.$$

   Since $\epsilon > 0$ was arbitrary, conclude that S has measure zero. □

2. Suppose that subset S of $\mathbf{R}$ is a set of measure zero.

   (a) Prove that every subset of S also has measure zero.

   (b) For fixed $x \in \mathbf{R}$, define $S + x = \{s + x : s \in S\}$. Prove that $S + x$ has measure zero.

   (c) For fixed finite $p > 0$, define $pS = \{ps : s \in S\}$. Prove that pS has measure zero.

   Solution: (a) Let $T \subset S \subset \mathbf{R}$ be given. Then any cover of S also covers T.

   If S has measure zero, then for every $\epsilon > 0$ there is a countable open cover of S, and therefore of T, by intervals with total length less than ϵ.

   Conclude that T satisfies the definition of a set of measure zero.

   (b) If $\{I_k : k = 1, 2, \ldots\}$ is a countable cover of S by open intervals $I_k = (a_k, b_k)$, then $\{I_k + x : k = 1, 2, \ldots\}$ is a countable cover of $S + x$ by open intervals $I_k + x = (a_k + x, b_k + x)$. Check that $|I_k + x| = |I_k|$ and conclude that $S + x$ has measure zero.

   (c) If $\{I_k : k = 1, 2, \ldots\}$ is a countable cover of S by open intervals $I_k = (a_k, b_k)$, then $\{pI_k : k = 1, 2, \ldots\}$ is a countable cover of pS by open intervals $pI_k = (pa_k, pb_k)$. Check that $|pI_k| = p|I_k|$ and conclude that pS has measure zero. □

3. Suppose that $f_n \to f$ and $g_n \to g$ both converge uniformly on S. Prove that $f_n - g_n$ converges uniformly to $f - g$ on S.

   Solution: Given $\epsilon > 0$, find $N_1 < \infty$ and $N_2 < \infty$ such that $n > N_1 \Rightarrow (\forall x \in S)\, |f_n(x) - f(x)| < \epsilon/2$ and $n > N_2 \Rightarrow (\forall x \in S)\, |g_n(x) - g(x)| < \epsilon/2$. Then if $n > N \stackrel{\text{def}}{=} \max\{N_1, N_2\}$, the triangle inequality assures us that $(\forall x \in S)\, \big|[f_n(x) - g_n(x)] - [f(x) - g(x)]\big| < \epsilon$. $\square$

4. Define $h_n(x) = f_n(x)g_n(x)$, where $f_n(x) = (1 + \frac{1}{n})x$ and

$$g_n(x) = \begin{cases} \frac{1}{n}, & \text{if } x = 0 \text{ or } x \text{ is irrational,} \\ b + \frac{1}{n}, & \text{if } x = \frac{a}{b} \text{ in lowest terms, with } b > 0 \text{ and } a \neq 0. \end{cases}$$

   (a) Prove that $\{f_n\}$ and $\{g_n\}$ converge uniformly on every bounded interval.

   (b) Prove that $\{h_n\}$ does **not** converge uniformly on any bounded interval.

   Solution: Suppose that the bounded interval is $[a, b]$.

   (a) For $\{f_n\}$, note that $|f_n(x) - f_m(x)| = |(\frac{1}{n} - \frac{1}{m})x| \le \max\{\frac{1}{n}, \frac{1}{m}\} \max\{|a|, |b|\} \to 0$ as $n, m \to \infty$. Hence $\{f_n\}$ satisfies the Cauchy condition uniformly on $[a, b]$. Likewise for $\{g_n\}$, note that $|g_n(x) - g_m(x)| = |(\frac{1}{n} - \frac{1}{m})| \le \max\{\frac{1}{n}, \frac{1}{m}\} \to 0$, so $\{g_n\}$ also satisfies the Cauchy condition uniformly on $[a, b]$. Hence both sequences of functions converge uniformly.

   (b) The problem is that each of the functions g_n is unbounded on every interval, since every interval contains rational numbers with arbitrarily large denominators. Suppose $x = a/b$ is a nonzero rational number expressed in lowest terms, with $b > 0$. Then $f_n(x)g_n(x) = (1 + \frac{1}{n})\frac{a}{b}(b + \frac{1}{n}) = a(1 + \frac{1}{n})(1 + \frac{1}{bn})$, and taking $m > n > 0$ without loss of generality we can estimate

$$\big|f_n(x)g_n(x) - f_m(x)g_m(x)\big| = |a| \left[(1 + \frac{1}{n})(1 + \frac{1}{bn}) - (1 + \frac{1}{m})(1 + \frac{1}{bm})\right] \ge |a|/n^2.$$

   Now suppose that we are given $\epsilon > 0$ and take some $N < \infty$. Since every interval with nonempty interior contains rational numbers with arbitrarily large numerators a, we can find a rational x in the interval with numerator $|a| > \epsilon(N + 100)^2$. Then the inequality implies that for $N < n, m < N + 100$, we have $\big|f_n(x)g_n(x) - f_m(x)g_m(x)\big| > \epsilon$. $\square$

5. Prove:

   (a) If $\{f_n\}$ and $\{g_n\}$ are both uniformly bounded sequences of functions on S, and if $f_n \to f$ and $g_n \to g$ both converge uniformly on S, then $f_n g_n \to fg$ will converge uniformly on S.

   (b) If $\{f_n\}$ and $\{1/g_n\}$ are both uniformly bounded sequences of functions on S, and if $f_n \to f$ and $g_n \to g$ both converge uniformly on S, then $f_n/g_n \to f/g$ will converge uniformly on S.

   Solution: (a) Suppose M is an upper bound for both $\{|f_n|\}$ and $\{|g_n|\}$. Then M is also an upper bound for $|f|$ and $|g|$. Now, given $\epsilon > 0$, choose $N < \infty$ so that $n > N$ implies $|f_n - f| < \frac{\epsilon}{2M}$ and $|g_n - g| < \frac{\epsilon}{2M}$. By the triangle inequality, $|f_n g_n - fg| \le |f_n - f||g_n| + |f||g_n - g| < \epsilon$.

   (b) The proof in (a) works for f_n/g_n. $\square$

6. Assume that

   i. $f_n \to f$ uniformly on S;

   ii. Each f_n is continuous on S;

   iii. The sequence $\{x_n\} \subset S$ converges to $a \in S$.

Prove that $f_n(x_n) \to f(a)$ as $n \to \infty$.

Solution: Let $\epsilon > 0$ be given. Since $f_n \to f$ uniformly, we can find $N_1 < \infty$ sufficiently large so that $n > N_1$ implies that $(\forall x \in S)|f_n(x) - f(x)| < \epsilon/2$. In particular, this means $|f_n(x_n) - f(x_n)| < \epsilon/2$.

Since f is the uniform limit of a sequence of continuous functions, it is continuous. Thus we can find $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon/2$. Since $x_n \to a$, we can find $N_2 < \infty$ sufficiently large so that $n > N_2 \Rightarrow |x_n - a| < \delta \Rightarrow |f(x_n) - f(a)| < \epsilon/2$. But then by the triangle inequality, if $n > \max\{N_1, N_2\}$,

$$|f_n(x_n) - f(a)| = |f_n(x_n) - f(x_n) + f(x_n) - f(a)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(a)| < \epsilon$$

□

7. Find counterexamples to show that each assumption i, ii, and iii is necessary in the previous problem.

Solution: Suppose we eliminate (i). Take $f_n(x) = x^n$ on $S = [0, 1]$ and take the sequence $x_n = 1 - \frac{1}{n}$. Then $f_n(x_n) = (1 - \frac{1}{n})^n \to \frac{1}{e}$ as $n \to \infty$, while $x_n \to a = 1$ and $f_n \to f$ where

$$f(x) = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{if } 0 \le x < 1. \end{cases}$$

Thus $f(a) = 1 \neq \frac{1}{e} = \lim_{n\to\infty} f_n(x_n)$.

Alternatively, if we eliminate (ii), let $f_n(x) = \frac{1}{x} + \frac{1}{n}$ when $x \neq 0$ with $f_n(0) \overset{\text{def}}{=} 0$ on $S = [0, 1]$. Then $f_n \to f$ uniformly on S, where $f(x) = 1/x$ when $x \neq 0$ and $f(0) = 0$, since $(\forall x \in S)|f_n(x) - f_m(x)| \le \max\{\frac{1}{n}, \frac{1}{m}\}$ and thus $\{f_n\}$ satisfies the uniform Cauchy condition for sequences of functions. However, if we take $x_n = \frac{1}{n}$, then $x_n \to a = 0$, and $f(a) = 0$ while $f_n(x_n) = n + \frac{1}{n} \to \infty$ as $n \to \infty$.

Without assumption (iii), the conclusion cannot even be stated, but we will find an example in which $\lim_{n\to\infty} f_n(x_n)$ fails to exist. Namely, take $f_n(x) = x$ on $S = [0, 1]$ and take $x_n = 1$ if n is odd and $x_n = 0$ if n is even. Then $f_n \to f(x) = x$ uniformly because it is a constant sequence of functions, but $f_n(x_n)$ alternates between 0 and 1 just like $\{x_n\}$ and therefore cannot have a limit. □

8. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, with $f(0) \overset{\text{def}}{=} 0$. Prove that $f^{(n)}(0)$ exists and equals 0 for each $n = 0, 1, 2, \ldots$.

Solution: First use induction on n to prove the following statement: $f^{(n)} = R_n(x)f(x)$, where $R_n(x)$ is a rational function. This is clearly true for $n = 0$. For $n = 1$, note that $\frac{d}{dx}f(x) = \frac{2}{x^3}e^{-1/x^2} = R_1(x)f(x)$, where $R_1(x) \overset{\text{def}}{=} \frac{2}{x^3}$ is a rational function. Now suppose the formula holds for $n = k$. Then $f^{(k+1)}(x) = \frac{d}{dx}f^{(k)}(x) = R_k'(x)f(x) + R_k(x)f'(x) = [R_k'(x) + R_k(x)R_1(x)]\, f(x)$, and $R_{k+1}(x) \overset{\text{def}}{=} R_k'(x) + R_k(x)R_1(x)$ is a rational function since differentiation and multiplication preserves that class.

Second, we claim that for any rational function $R(x)$, we have $\lim_{x\to 0} R(x)f(x) = 0$. If $R(x)$ is continuous at 0, the result follows from the product limit theorem since $f(x) \to 0$ as $x \to 0$. If $R(x)$ is discontinuous at 0, then its denominator polynomial must have a root at 0 so we can write $R(x) = Q(x)x^{-k}$ where $k > 0$ and $Q(x)$ is a rational function which is continuous at $x = 0$. Hence it suffices to show that $x^{-k}f(x) \to 0$ as $x \to 0$ for every $k > 0$. Without loss of generality, it suffices to prove this for $x \to 0+$. But

$$\log\left[x^{-k}f(x)\right] = -k\log x - \frac{1}{x^2} = \frac{kx^2\log x + 1}{-x^2}.$$

The numerator tends to 1 as $x \to 0+$, since $x^2 \log x \to 0$ as $x \to 0+$ which was shown using L'Hôpital's theorem. The denominator tends to $0-$ as $x \to 0$. Thus the ratio tends to $-\infty$ and therefore $x^{-k}f(x) \to 0$ as $x \to 0$. □

9. Suppose $f(x)$ is represented by the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 = a_1 = 3$ while $a_n = 1$ for $n \geq 2$. What is the power series for $1/f(x)$, and its radius of convergence?

Solution: The given power series has a radius of convergence of 1, and represents the function $f(x) = \frac{3-2x^2}{1-x}$, which we discover by rewriting the series as

$$\sum_{n=0}^{\infty} a_n x^n = 2 + 2x + \sum_{n=0}^{\infty} x^n = 2 + 2x + \frac{1}{1-x} = \frac{3-2x^2}{1-x}.$$

The reciprocal is $1/f(x) = \frac{1-x}{3-2x^2}$, which has power series

$$\frac{1-x}{3}\left(\frac{1}{1-\frac{2x^2}{3}}\right) = \frac{1-x}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}x^2\right)^n = \sum_{n=0}^{\infty}\left[\frac{1}{3}\left(\frac{2}{3}\right)^n x^{2n} - \frac{1}{3}\left(\frac{2}{3}\right)^n x^{2n+1}\right] = \sum_{n=0}^{\infty} b_n x^n,$$

where $b_{2n} = \frac{1}{3}\left(\frac{2}{3}\right)^n$ (even terms) while $b_{2n+1} = -\frac{1}{3}\left(\frac{2}{3}\right)^n$ (odd terms). The reciprocal series has radius of convergence $\sqrt{3/2}$, by the ratio test. □

10. Let $f_n(x)$ be the real-valued function defined on $[0,1]$ by the formula

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \leq x < 2^{-n}; \\ 2^{n/2}, & \text{if } 2^{-n} \leq x \leq 2^{1-n}; \\ 0, & \text{if } 2^{1-n} < x \leq 1, \end{cases}$$

for $n = 1, 2, \ldots$. Prove that $\{f_n\}$ converges pointwise to 0 on $[0,1]$, but $\text{l.i.m.}_{n\to\infty} f_n \neq 0$.

Solution: To show pointwise convergence, suppose that $x \in [0,1]$. If $x = 0$, then $f_n(x) = 0$ for all $n = 1, 2, \ldots$, so that $\lim_{n\to\infty} f_n(0) = 0$. If $x > 0$, then $f_n(x) = 0$ for all n satisfying $2^{1-n} < x$, which is all $n > 1 - \log_2 x$. Hence again, $\lim_{n\to\infty} f_n(x) = 0$.

Now suppose $\text{l.i.m.}_{n\to\infty} f_n = 0$; then for $\|g\| \stackrel{\text{def}}{=} \int_0^1 |g(x)|^2\,dx$ we would have $\|f_n\| \to 0$ as $n \to \infty$. But elementary calculus allows us to evaluate $\|f_n\| = 1$ for all $n = 1, 2, \ldots$, contradicting our assumption. Thus either the limit in mean fails to exist, or it exists and equals some function other than 0. But since the pointwise limit is the 0 function, there are no other candidates, hence the limit in mean does not exist. □