Ma 4121: Introduction to Lebesgue Integration
Solutions to [Revised] Homework Assignment 2

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Due Thursday, February 14th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

NOTE: only exercises 1, 2, 8b and 9 are due for this HW set.

1. Suppose that $\{f_n\}$ and $\{g_n\}$ are increasing sequences of functions on an interval I, and put $u_n = \max(f_n, g_n)$. Prove that if $f_n \nearrow f$ *a.e.* on I, and $g_n \nearrow g$ *a.e.* on I, then $u_n \nearrow \max(f, g)$ *a.e.* on I.

   Solution: Let $D_f \subset I$ be the set of points x where $f_n(x)$ does not converge to $f(x)$, and let $D_g \subset I$ be the set of points x where $g_n(x)$ does not converge to $g(x)$. Then $D \stackrel{\text{def}}{=} D_f \cup D_g$ has measure zero. Now let $\epsilon > 0$ be given and fix $x \in I - D$. Then there is an $N < \infty$ such that for each $n \geq N$ we have $f_n(x) \leq f(x) < f_n(x) + \epsilon$ and $g_n(x) \leq g(x) < g_n(x) + \epsilon$. But then $\max(f_n(x), g_n(x)) \leq \max(f(x), g(x)) < \max(f_n(x), g_n(x)) + \epsilon$. Since $\epsilon > 0$ was arbitrary, we must have $\lim_{n\to\infty} \max(f_n(x), g_n(x)) = \max(f(x), g(x))$. Since this holds for all $x \in I - D$, it holds *a.e.* on I. □

2. Let $I = [0, 1]$. Find a function $f \in U(I)$ such that $-f \notin U(I)$.

   Solution: Let $\{r_n : n = 1, 2, 3, \ldots\}$ be an enumeration of the rationals, and put $I_n \stackrel{\text{def}}{=} (r_n - 8^{-n}, r_n + 8^{-n}) \cap I$. Then I_n is an interval for all n, and its length is no more than $2 \times 8^{-n} = 2^{1-3n}$.

   Now define a function f on I and a sequence of step functions $\{f_n\}$ on I by the formulas

   $$f_n(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } x \in I_1 \cup \ldots \cup I_n, \\ 0, & \text{otherwise.} \end{cases} \qquad f(x) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } x \in I_n \text{ for some } n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Then $f_n \nearrow f$ on I, so f belongs to $U(I)$.

   However, suppose that $s_n \in S(I)$ and $s_n(x) \leq -f(x)$ *a.e.* on I. Since every open subinterval of I contains a neighborhood of some rational point r_n, the step function s_n must satisfy $s_n(x) \leq -1$ *a.e.* on I. Thus $\int_I s_n \leq -1$, so that $\lim \int_I s_n \leq -1$. However, since $\int_I f = \sum |I_n| \leq 2 \times \frac{1}{8}(1 - \frac{1}{8})^{-1} = 2/7$, and $-f \in L(I)$, we must have $\int_I(-f) \geq -2/7 > -1$. Thus no monotone increasing sequence of step functions can converge to $-f$. □

3. Suppose that $\{f_n\} \subset L(I)$ satisfies $(\forall n) f_n(x) \geq 0$ *a.e.* on I, and $f_n \to f$ *a.e.* on I, and $(\exists A < \infty)(\forall n) \int_I f_n \leq A$. Prove that the limit function f belongs to $L(I)$ and that $\int_I f \leq A$. (this is called *Fatou's Lemma*).

   —Postponed to next HW.

4. Find, with proof, all $p \in \mathbf{R}$ for which the Lebesgue integral $\int_0^\infty x^p \sin(x^2)\, dx$ exists.

   —Postponed to next HW.

5. Prove that the following Lebesgue integrals exist:

$$\int_0^1 (x \log x)^2 \, dx, \qquad \int_0^1 \log x \log(1-x)^2 \, dx, \qquad \int_0^1 \frac{\sqrt{1-x}}{\log x} \, dx.$$

—*Postponed to next HW.*

6. For each of the Lebesgue integrals and intervals I below, determine with proof the set S of values $s \in \mathbf{R}$ for which it must exist for every function $f \in L(I)$. For each s not in S, find a bounded continuous f for which the Lebesgue integral fails to exist.

$$\int_0^1 f(x) \cos(2\pi s x) \, dx, \qquad \int_0^\infty f(x) e^{sx} \, dx, \qquad \int_0^\infty \frac{f(x)}{x^2 + s^2} \, dx.$$

—*Postponed to next HW.*

7. Suppose that f is continuous on $[0,1]$, $f(0) = 0$, and $f'(0)$ exists. Prove that the Lebesgue integral $\int_0^1 f(x) x^{-3/2} \, dx$ exists.

—*Postponed to next HW.*

8. Suppose that $f \in L([0,1])$ and put $g_n(x) \stackrel{\text{def}}{=} f(x)\sin(nx)$ for integers n.

(a) Prove that g_n also belongs to $L([0,1])$.

(b) Prove that $\int_0^1 g_n \to 0$ as $|n| \to \infty$. (Hint: first prove the result for step functions.)

Solution: Denote $[0,1]$ by I.

(a)

—*Postponed to next HW.*

(b) Using the hint, suppose $f \in S(I)$ and put $f(x) = c_k$ if $x_{k-1} \le x < x_k$ and $1 \le k \le m$. Evaluate the integral of g_n with this f:

$$\int_0^1 f(x) \sin(nx) \, dx = \sum_{k=1}^m c_k \frac{-cos(nx)}{n} \bigg|_{x_{k-1}}^{x_k} = \frac{1}{n} \sum_{k=1}^m c_k \left[\cos(nx_{k-1}) - \cos(nx_k) \right]$$

The sum is finite and fixed for all n, hence the right-hand side tends to 0 as $n \to \infty$.

Now suppose only that $f \in L(I)$, but let $\epsilon > 0$ be given. Then by Theorem 10.19 there is a step function s with the property that $\int_I |f - s| < \epsilon/2$. Now choose $N < \infty$ so large that $n \ge N \Rightarrow \int_I s(x) \sin(nx) \, dx < \epsilon/2$, and observe that $\int_I |f(x) - s(x)| |\sin(nx)| \, dx \le \int_I |f(x) - s(x)| \, dx < \epsilon/2$. Then the triangle inequality yields $\| \int_I f(x) \sin(nx) \, dx \| < \epsilon$. Since $\epsilon > 0$ was arbitrary, the integral must tend to 0 as $n \to \infty$. □

9. Suppose that $I = \mathbf{R}$ and $f \in L(I)$. Put $f_y(x) \stackrel{\text{def}}{=} f(x - y)$ for $y \in \mathbf{R}$. Prove that $\int_I |f_y - f| \to 0$ as $y \to 0$. (Hint hint: see previous hint.)

Solution: First prove the result for a step function s satisfying $s(x) = c_k$ if $x_{k-1} \le x < x_k$ and $1 \le k \le m$ with $-x_0 = x_m = N < \infty$. Let $d = \min\{x_k - x_{k-1} : 1 \le k \le m\}$, let $M = \max\{|c_k - c_{k-1}| : 1 \le k \le m\}$, and suppose that $\epsilon > 0$ is given. Then $|y| < \frac{\epsilon d}{2NM}$ implies that $\int_I |s_y - s| < \epsilon$, and so the integral must vanish as $n \to \infty$.

Again, let $\epsilon > 0$ be chosen. Write $f = s + g$ where $s \in S(\mathbf{R})$ and $g \in L(\mathbf{R})$ with $\int_{\mathbf{R}} |g| < \epsilon/4$, as in Theorem 10.19. There is some $N < \infty$ with $s(x) = 0$ if $|x| > N$, since all step functions vanish outside a bounded interval, so choose $y_0 > 0$ by the criterion in the first paragraph so that $\int_{\mathbf{R}} |s_y - s| < \epsilon/2$. Then by the triangle inequality,

$$\int_{\mathbf{R}} |f_y - f| \leq \int_{\mathbf{R}} |s_y - s| + \int_{\mathbf{R}} |f_y - s_y| + \int_{\mathbf{R}} |f - s| < \epsilon.$$

$\square$

10. If f is Lebesgue-integrable on an open interval I and if f' exists *a.e.* on I, prove that f' is measurable on I.

    —*Postponed to next HW.*