Ma 4121: Introduction to Lebesgue Integration
Solutions to Homework Assignment 3

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Due Thursday, February 28th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Let $\sim$ denote the relation on $\mathbf{R}$ defined by

$$x \sim y \iff x - y \in \mathbf{Q}$$

Prove that $\sim$ is an equivalence relation, namely:

(i) $(\forall x \in \mathbf{R}) x \sim x$

(ii) $(\forall x, y \in \mathbf{R}) x \sim y \iff y \sim x$

(iii) $(\forall x, y, z \in \mathbf{R})(x \sim y \text{ and } y \sim z) \Rightarrow x \sim z$

Solution: For (i), note that $x - x = 0 \in \mathbf{Q}$.

For (ii), note that $x - y \in \mathbf{Q} \Rightarrow y - x = -(x - y) \in \mathbf{Q}$, since $\mathbf{Q}$ is closed under negation.

For (iii), note that $x - y \in \mathbf{Q}$ and $y - z \in \mathbf{Q}$ implies $(x - y) + (y - z) = x - z \in \mathbf{Q}$, since $\mathbf{Q}$ is closed under addition. □

2. Given $x \in \mathbf{R}$, define the equivalence class $[x] = \{y \in \mathbf{R} : x \sim y\}$, where $\sim$ is the equivalence relation of exercise 1 above.

(a) Prove that $[x]$ is countably infinite for every $x \in \mathbf{R}$.

(b) Prove that the number of distinct equivalent classes is uncountable.

Solution: (a) Let $\{r_1, r_2, \ldots\}$ be an enumeration of $\mathbf{Q}$. If $y \in [x]$, then $y = x - r_k$ for exactly one $k \in \mathbf{Z}^+$. Hence there is a one-to-one correspondence between elements of $[x]$ and a subset of the countable set $\mathbf{Z}^+$, so $[x]$ is countable.

To show that $[x]$ is infinite, observe that it contains the infinite set $\{x + 1/n : n \in \mathbf{Z}^+\}$.

(b) For $x, y \in \mathbf{R}$, either $[x] = [y]$ (if $x \sim y$) or $[x] \cap [y] = \emptyset$ (if $x - y$ is irrational). Also,

$$\mathbf{R} \subset \bigcup_{x \in \mathbf{R}} [x],$$

and since each $[x]$ is countable by part (a), there must be uncountably many distinct equivalent classes, or else the union would be a countable set with an uncountable subset, which is impossible. □

3. Suppose that $\{f_n\} \subset L(I)$ satisfies $(\forall n) f_n(x) \geq 0$ *a.e.* on I, and $f_n \to f$ *a.e.* on I, and $(\exists A < \infty)(\forall n) \int_I f_n \leq A$. Prove that the limit function f belongs to $L(I)$ and that $\int_I f \leq A$. (this is called *Fatou's Lemma*).

Solution: Define $g_n \in L(I)$ by $g_n(x) = \inf\{f_n(x), f_{n+1}(x), \ldots\}$. Then $0 \le g_n(x) \le f_m(x)$ for all $m \ge n$ and almost all $x \in I$. This implies (a) that $0 \le \int_I g_n \le \int_I f_m \le A$ for all n, and also (b) that $0 \le g_n(x) \le f(x)$ *a.e.* on I for all m. From (a), the Levi theorem for increasing sequences of functions in $L(I)$ implies that there is some function $g \in L(I)$ such that $g_n \to g \le f$ *a.e.* on I and $\lim_{n\to\infty} \int_I g_n = \int_I g \le \int_I f$.

Now fix $x \in I$ to be a point where $\{f_n(x)\}$ converges. Then for every $\epsilon > 0$ there is some $N < \infty$ such that $n \ge N \Rightarrow f(x) - \epsilon \le g_n(x) \le f(x)$. Thus $g_n \to f$ *a.e.* on I, so $\int_I f = \lim_{n\to\infty} \int_I g_n$. Conclusion (b) from the previous paragraph implies that the limit is no larger than A. □

4. Find, with proof, all $p \in \mathbf{R}$ for which the Lebesgue integral $\int_0^\infty x^p \sin(x^2)\, dx$ exists.

Solution: Write $f(x) = x^p \sin(x^2)$. By Theorem 10.18, $f \in L([0,\infty])$ iff $f \in L([0,1])$ and $f \in L([1,\infty])$,

$f \in L[0,1]$: Note that $\sin(1)x^2 \le \sin(x^2) \le x^2$ for any $x \in [0,1]$, so that $f \in L([0,1])$ iff $x^{p+2} \in L([0,1])$. But x^{p+2} is a continuous function on the bounded interval $(0,1]$, so we can use improper Riemann integration to conclude that $f \in L([0,1])$ iff $p > -3$.

$f \in L[1,\infty)$: Since $|f(x)| \le x^p$ on $[1,\infty)$ we know that $p < -1 \Rightarrow f \in L([1,\infty))$. For the converse, define

$$s(x) = \begin{cases} \frac{1}{2}[(2n+\frac{1}{3})\pi]^{p/2}, & \text{if } [(2n+\frac{1}{3})\pi]^{1/2} \le x \le [(2n+\frac{2}{3})\pi]^{1/2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then s is measurable and $0 \le s \le |f|$ on $[1,\infty)$, so that $f \in L([1,\infty)) \Rightarrow s \in L([1,\infty))$. However, since s has countably many discontinuities, it is Riemann integrable on any bounded subinterval of $[0,\infty)$, so its integrability depends only on the existence of $\lim_{b\to\infty} \int_1^b s(x)\, dx$. The latter is just the sum of the positive infinite series:

$$\sum_{n=1}^{\infty} \frac{1}{2}[(2n+\frac{1}{3})\pi]^{p/2} \left([(2n+\frac{2}{3})\pi]^{1/2} - [(2n+\frac{1}{3})\pi]^{1/2} \right).$$

The difference may be estimated using the mean value theorem; it lies between $\frac{1}{2}[(2n+\frac{1}{3})\pi]^{-\frac{1}{2}}\frac{\pi}{3}$ and $\frac{1}{2}[(2n+\frac{2}{3})\pi]^{-\frac{1}{2}}\frac{\pi}{3}$. The series thus converges by the comparison test iff

$$\sum_{n=1}^{\infty} n^{\frac{p}{2}} n^{-\frac{1}{2}} = \sum_{n=1}^{\infty} n^{\frac{p-1}{2}}$$

converges, which happens iff $(p-1)/2 < -1$, namely iff $p < -1$.

Thus $f \in L([0,\infty))$ if and only if $f \in L([0,1])$ and $f \in L([1,\infty))$, if and only if $p \in (-3,-1)$. □

5. Prove that the following Lebesgue integrals exist:

$$\int_0^1 (x\log x)^2\, dx, \qquad \int_0^1 \log x \log(1-x)^2\, dx, \qquad \int_0^1 \frac{\sqrt{1-x}}{\log x}\, dx.$$

Solution: The function $(x\log x)^2$ is continuous, hence measurable, and bounded between 0 and e^{-2} on $(0,1)$. Thus it is Lebesgue integrable on $[0,1]$ by Corollary 2 of Theorem 10.35 in the text.

For the measurable function $\log x \log(1-x)^2$, the identities $2(x-1) \le \log x \le x-1$ (for $\frac{1}{2} < x < 1$) and $x \le |\log(1-x)^2| \le 2x$ (for $0 < x < \frac{1}{2}$) give the following estimate:

$$|\log x \log(1-x)^2| \le 2x|\log x| + 4(1-x)|\log(1-x)|.$$

The right hand side is bounded on $(0,1)$, so Corollary 2 may be used again.

Using the same estimates, the continuous and hence measurable function $|\sqrt{1-x}/\log x|$ can be bounded on $(0,1)$ by $1+(1-x)^{-1/2}$, which is improperly Riemann integrable:

$$\int_0^1 \left[1+(1-x)^{-1/2}\right]\,dx = 1 + \lim_{b\to 1-} [-2(1-x)^{1/2}]\big|_0^b = 1+2<\infty.$$

Since the interval $(0,1)$ is bounded, the function is Lebesgue integrable there too. □

6. For each of the Lebesgue integrals and intervals I below, determine with proof the set S of values $s \in \mathbf{R}$ for which it must exist for every function $f \in L(I)$. For each s not in S, find a bounded continuous f for which the Lebesgue integral fails to exist.

$$\int_0^1 f(x)\cos(2\pi s x)\,dx, \qquad \int_0^\infty f(x)e^{sx}\,dx, \qquad \int_0^\infty \frac{f(x)}{x^2+s^2}\,dx.$$

Solution: In the first integral, $I=[0,1]$. Since $|\cos(2\pi s x)| \le 1$ for all $s \in \mathbf{R}$ and all $x \in I$, we know by Theorem 10.35 that $f \in L(I) \Rightarrow f(x)\cos(2\pi s x) \in L(I)$ for every $s \in \mathbf{R}$. Thus $S=\mathbf{R}$ and there is nothing further to prove.

In the second integral, $I=[0,+\infty)$. If $s \le 0$, then $0<e^{sx}\le 1$ for all $x \in I$, so again by Theorem 10.35 we have $f \in L(I) \Rightarrow f(x)e^{sx} \in L(I)$. But if $s>0$, then the bounded continuous $L(I)$ function $f(x)=e^{-sx/2}$ yields $f(x)e^{sx}=e^{sx/2} \notin L(I)$, since the improper Riemann integral diverges. Thus $S=(-\infty,0]$.

In the third integral, we have $I=[0,+\infty)$ once more. If $s \ne 0$, then $1/(x^2+s^2) \le 1/s^2 < \infty$, so $f \in L(I) \Rightarrow f(x)/(x^2+s^2) \in L(I)$ as before. However, if $s=0$, then the following bounded continuous $L(I)$ function ceases to belong to $L(I)$ upon multiplication by $1/x^2$:

$$f(x) = \begin{cases} 1-x, & \text{if } 0 \le x \le 1; \\ 0, & \text{if } x>1. \end{cases}$$

That is because $0 < \frac{1}{2}x^{-2} < \frac{1-x}{x^2}$ for all $0<x<\frac{1}{2}$, and the measurable function $1/x^2$ is not in $L([0,\frac{1}{2}])$, so it cannot be dominated by a Lebesgue integrable function on $[0,\frac{1}{2}]$, so $\frac{1-x}{x^2}$ cannot be in $L([0,\frac{1}{2}])$ and hence cannot be in $L([0,+\infty))$. Thus $S=\mathbf{R}-\{0\}$. □

7. Suppose that f is continuous on $[0,1]$, $f(0)=0$, and $f'(0)$ exists. Prove that the Lebesgue integral $\int_0^1 f(x)x^{-3/2}\,dx$ exists.

Solution: Since f is continuous it is bounded on $[0,1]$. Let $g(x)=[f(x)-f(0)]/(x-0)=f(x)/x$ if $0<x\le 1$ and put $g(0) \stackrel{\text{def}}{=} f'(0)$. Then g is continuous and hence bounded on $[0,1]$, say by $M<\infty$. Finally, note that $|f(x)x^{-3/2}| = |g(x)x^{-1/2}| \le M|x^{-1/2}|$. But $M|x^{-1/2}|$ is positive and improperly Riemann integrable on $[0,1]$, so it is in $L([0,1])$, and $f(x)x^{-3/2}$ is measurable, so Theorem 10.35 implies that the Lebesgue integral of $f(x)x^{-3/2}$ exists. □

8. Suppose that $f \in L([0,1])$ and put $g_n(x) \stackrel{\text{def}}{=} f(x)\sin(nx)$ for integers n.

(a) Prove that g_n also belongs to $L([0,1])$.

(b) Prove that $\int_0^1 g_n \to 0$ as $|n| \to \infty$. (Hint: first prove the result for step functions.)

Solution: Denote $[0,1]$ by I.

(a) Since $\sin nx$ is a continuous function it is measurable on I, hence $g_n \in M(I)$. Since $|\sin\theta| \le 1$ for all real numbers θ, we have $|g_n| \le |f|$. Thus $g_n \in L(I)$ by the corollary to Theorem 10.35.

(b) Using the hint, suppose $f \in S(I)$ and put $f(x) = c_k$ if $x_{k-1} \le x < x_k$ and $1 \le k \le m$. Evaluate the integral of g_n with this f:

$$\int_0^1 f(x)\sin(nx)\,dx = \sum_{k=1}^{m} c_k \frac{-\cos(nx)}{n}\bigg|_{x_{k-1}}^{x_k} = \frac{1}{n}\sum_{k=1}^{m} c_k\left[\cos(nx_{k-1}) - \cos(nx_k)\right]$$

The sum is finite and fixed for all n, hence the right-hand side tends to 0 as $n \to \infty$.

Now suppose only that $f \in L(I)$, but let $\epsilon > 0$ be given. Then by Theorem 10.19 there is a step function s with the property that $\int_I |f - s| < \epsilon/2$. Now choose $N < \infty$ so large that $n \ge N \Rightarrow \int_I s(x)\sin(nx)\,dx < \epsilon/2$, and observe that $\int_I |f(x) - s(x)||\sin(nx)|\,dx \le \int_I |f(x) - s(x)|\,dx < \epsilon/2$. Then the triangle inequality yields $\| \int_I f(x)\sin(nx)\,dx\| < \epsilon$. Since $\epsilon > 0$ was arbitrary, the integral must tend to 0 as $n \to \infty$. □

9. Put $I = [0, 1]$. Suppose that f is continuous on I with $f(0) = 0$, and that $f'(0)$ exists and is finite. (Here we mean the one-sided derivative

$$f'(0) \stackrel{\text{def}}{=} \lim_{h\to 0+} \frac{f(h) - f(0)}{h}.)$$

Prove that $g(x) \stackrel{\text{def}}{=} f(x)/x^{3/2}$ belongs to $L(I)$.

Solution: First note that $f \in M(I)$ because it is continuous. Similarly, f is bounded because continous functions are bounded on compact intervals such as I. This bound is a constant function which is evidently in $S(I)$, hence is in $L(I)$. Applying Lebesgue's dominated convergence theorem shows that $f \in L(I)$.

Conclude that $|f| \in L(I)$ as well,

Next, use Taylor's theorem on f at $x = 0$ to write

$$f(h) = f(0) + hf'(0) + o(h), \qquad \text{as } h \to 0+.$$

Thus we may choose $0 < \epsilon < 1$ such that

$$0 < h < \epsilon \Rightarrow |f(h)| = |f(h) - f(0)| \le (|f'(0)| + 1)h$$

Now write $I = [0, \epsilon] \cup [\epsilon, 1] \stackrel{\text{def}}{=} I_0 \cup I_1$. Then $g \in L(I_1)$ because it is continuous and bounded on I_1 (by the function $\epsilon^{-3/2}|f| \in L(I)$), and $g \in L(I_0)$ because it is continuous (except at 0), hence measurable, and dominated by the function $(|f'(0)| + 1)x/x^{3/2} = (|f'(0)| + 1)/x^{1/2}$, which is improperly Riemann integrable over I_0 and thus belongs to $L(I_0)$.

Conclude that $g \in L(I) = L(I_0 \cup I_1)$. □

10. If f is Lebesgue-integrable on an open interval I and if f' exists *a.e.* on I, prove that f' is measurable on I.

Solution: Let $\{h_n : n = 1, 2, \ldots\}$ be a sequence of real numbers converging to 0, and define the functions $\{g_n\}$ by $g_n(x) = [f(x + h_n) - f(x)]/h_n$. Then $g_n \to f'$ *a.e.* on I, and each g_n is measurable since it is a linear combination of measurable functions. But then f' is measurable by Theorem 10.37. □