Ma 4121: Introduction to Lebesgue Integration
Solutions to Homework Assignment 4

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Due Thursday, March 28th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Find a counterexample to show that the union of two σ-algebras for a set X need not be a σ-algebra on X. (Hint: it is enough to consider a three-point set X.)

   Solution: In fact we will show by counterexample that the union of two algebras is not necessarily an algebra.

   Let $X = \{1, 2, 3\}$ and put $A = \{X, \emptyset, \{1\}, \{2, 3\}\}$ and $B = \{X, \emptyset, \{2\}, \{1, 3\}\}$. Then A and B are both algebras, hence σ-algebras because X is finite, but

   $$A \cup B = \{X, \emptyset, \{1\}, \{2, 3\}, \{2\}, \{1, 3\}\},$$

   which is not an algebra, hence not a σ-slgebra, because it does not contain $\{1, 2\} = \{1\} \cup \{2\}$. $\square$

2. Show that the Borel sets $\mathcal{B}(\mathbf{R})$ are generated by the compact subsets of $\mathbf{R}$.

   Solution: Write $\mathcal{U}^\sigma$ for the σ-algebra generated by a collection $\mathcal{U}$ of subsets of X, and note that $\mathcal{U} \subset \mathcal{V} \subset 2^X$ implies $\mathcal{U}^\sigma \subset \mathcal{V}^\sigma$. Also note that if $\mathcal{U}$ is itself a σ-algebra, then $\mathcal{U}^\sigma = \mathcal{U}$.

   Let $\mathcal{K}^\sigma$ be the σ-algebra generated by the collection $\mathcal{K}$ of compact subsets of $\mathbf{R}$. Then $\mathcal{K}$ contains all compact intervals, which generate $\mathcal{B}(\mathbf{R})$, so $\mathcal{B}(\mathbf{R}) \subset \mathcal{K}^\sigma$.

   Now suppose $K \subset \mathbf{R}$ is a compact set, so $K \in \mathcal{K}$. Then K is closed, so K^c is open, so by the structure theorem for open sets in $\mathbf{R}$, $K^c = \bigcup_n I_n$ where $\{I_n\}$ is a countable collection of pairwise disjoint open intervals. But $I_n \in \mathcal{B}(\mathbf{R})$ for all n, so $K^c \in \mathcal{B}(\mathbf{R})$, so $K \in \mathcal{B}(\mathbf{R})$. Thus $\mathcal{K}^\sigma \subset \mathcal{B}(\mathbf{R})$.

   Conclude that $\mathcal{B}(\mathbf{R}) = \mathcal{K}^\sigma$. $\square$

3. Let $(X, \mathcal{A})$ be a measurable space and suppose $\mu : \mathcal{A} \to [0, +\infty]$ is a countably additive function on the σ-algebra $\mathcal{A}$.

   (a) Show that if μ satisfies $\mu(A) < \infty$ for some $A \in \mathcal{A}$, then $\mu(\emptyset) = 0$. (This implies that μ is a measure.)

   (b) Find an example μ for which $\mu(\emptyset) \neq 0$. (Thus the first property of a measure does not follow from countable additivity and non-negativity.)

Solution: (a) Let A satisfy $\mu(A) = c < \infty$. Then $A = A \cup \emptyset$ is a disjoint union representing A, so by countable additivity,

$$\mu(A) = \mu(A) + \mu(\emptyset) \;\Rightarrow\; c = c + \mu(\emptyset) \;\Rightarrow\; \mu(\emptyset) = 0.$$

(b) Let X be any set, put $\mathcal{A} = 2^X$, and define $\mu(A) = +\infty$ for any nonempty $A \subset X$, with $\mu(\emptyset) = 1$. $\square$

4. Let $(X, \mathcal{A})$ be a measurable space. Say that a sequence of measures $\{\mu_n : n = 1, 2, \ldots\}$ is *increasing* iff

$$(\forall A \in \mathcal{A})(\forall n)\ \mu_{n+1}(A) \geq \mu_n(A)$$

(a) Show that if $\{\mu_n\}$ is an increasing sequence of measures, then

$$\mu(A) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mu_n(A)$$

defines a measure on $(X, \mathcal{A})$.

(b) Let $\{\mu_n : n = 1, 2, \ldots\}$ be any sequence of measures on $(X, \mathcal{A})$. Prove that

$$\mu(A) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \mu_n(A)$$

defines a measure on $(X, \mathcal{A})$.

Solution: (a) Clearly $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

Also,

$$\mu(\emptyset) = \lim_{n\to\infty} \mu_n(\emptyset) = \lim_{n\to\infty} 0 = 0.$$

To check countable additivity, suppose that $\{A_k\} \subset \mathcal{A}$ is a pairwise disjoint sequence. Then

$$(\forall n)\ \mu_n(A) = \sum_k \mu_n(A_k), \qquad A \stackrel{\text{def}}{=} \cup_k A_k \in \mathcal{A}.$$

since, for every n, μ_n is a measure.

Since $\{\mu_n\}$ increases, the nondecreasing sequence $\{\mu_n(A)\} \subset \mathbf{R}$ has a limit $\mu(A)$ in $[0, +\infty]$.

Suppose first that $\mu(A) = +\infty$. But then for each $M \in \mathbf{R}$ there is some $N = N(M)$ such that $n \geq N \Rightarrow \mu_n(A) > M$. But then there is some $K \in \mathbf{Z}^+$ such that

$$\sum_{k=1}^{K} \mu_N(A_k) > M - 1.$$

Since $\{\mu_n\}$ increases, using bigger n does not decrease the sum, so

$$n \geq N \;\Rightarrow\; \sum_{k=1}^{K} \mu_n(A_k) > M - 1, \quad \Rightarrow\; \sum_{k=1}^{K} \mu(A_k) > M - 1.$$

Adding more nonnegative terms also does not reduce the sum, so $\sum_{k=1}^{\infty} \mu(A_k) > M - 1$. Since M was arbitrary, conclude that $\sum_{k=1}^{\infty} \mu(A_k) = +\infty = \mu(A)$.

For the second case, $\mu(A) < \infty$, begin by noting that

$$(\forall n)\mu_n(A) = \sum_k \mu_n(A_k), \quad \Rightarrow \ (\forall n)\mu_n(A) \leq \sum_k \mu(A_k),$$

since $(\forall n)(\forall k)\mu(A_k) \geq \mu_n(A_k)$. Conclude that $\mu(A) \leq \sum_k \mu(A_k)$.

Finish by proving the reverse inequality, first noting that $0 \leq \mu_n(A_k) < \infty$ for all n, k. Since $\{\mu_n\} \nearrow$ and each μ_n is a measure, this implies

$$(\forall n)\ \mu(A) \geq \sum_{k=1}^{\infty} \mu_n(A_k). \qquad \Rightarrow \ (\forall n)(\forall K)\ \mu(A) \geq \sum_{k=1}^{K} \mu_n(A_k).$$

Now let $\epsilon > 0$ and $K < \infty$ be given. Choose $N = N(\epsilon, K)$ such that $n \geq N \Rightarrow \mu_n(A_k) \geq \mu(A_k) - \epsilon 2^{-k}$ for each $k = 1, 2, \ldots, K$. Then

$$\sum_{k=1}^{K} \mu_N(A_k) \geq \sum_{k=1}^{K} \mu(A_k) - \epsilon,$$

which combined with the previous inequality gives

$$(\forall K)\ \mu(A) \geq \sum_{k=1}^{K} \mu(A_k) - \epsilon, \qquad \Rightarrow \ \mu(A) \geq \sum_{k=1}^{\infty} \mu(A_k) - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, conclude that $\mu(A) \geq \sum_{k=1}^{\infty} \mu(A_k)$. Hence equality holds.

(b) Finite sums of measures are obviously measures, and

$$\nu_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} \mu_i$$

gives an increasing sequence of measures. Hence by part a, $\lim_{n\to\infty} \nu_n = \sum_{n=1}^{\infty} \mu_n$ is a measure. □

5. Let $(X, \mathcal{A})$ be a measurable space and define the function $\delta_x : \mathcal{A} \to [0, +\infty]$ by

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Prove that δ_x is a measure.

Solution: Evidently $\delta_x(A) \geq 0$ and $\delta_x(\emptyset) = 0$.

To check countable additivity, let $A = \cup_i A_i$ be a countable union of disjoint sets. There are two cases to consider:

-If $x \in A$, then $\delta_x(A) = 1$, but also $(\exists! i) x \in A_i$, so $\sum_i \delta_x(A_i) = 1$.

-Else $x \notin A$, so $\delta_x(A) = 0$, but also $(\forall) x \notin A_i$, so $\sum_i \delta_x(A_i) = 0$. □

6. Let $\{x_n : n = 1, 2, \dots\} \subset \mathbf{R}$ be a sequence of points and define a measure μ on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ by

$$\mu = \sum_n \delta_{x_n},$$

where δ_x is the measure defined in exercise 5.

(a) Show that μ assigns finite values to bounded intervals if and only if $|x_n| \to +\infty$ as $n \to \infty$.

(b) For what sequences is this μ a σ-finite measure?

Solution:

(a) Suppose $|x_n| \to +\infty$. Given a bounded interval I, there is some $M \in \mathbf{R}^+$ such that $I \subset B(0, M)$. There is some $N \in \mathbf{Z}^+$ such that $n \geq N \Rightarrow |x_n| > M \Rightarrow x_n \notin I$. But then $\mu(I) < N < \infty$.

Otherwise, there is some $M \in \mathbf{R}^+$ such that $\{x_n\} \cap B(0, M)$ is an infinite subsequence. But then by the Bolzano-Weierstrass theorem, this bounded infinite subsequence has an accumulation point, which we may call z. Any open interval containing z also contains infinitely many points of $\{x_n\}$, so in particular the bounded interval $I = (z-1, z+1)$ satisfies $\mu(I) = +\infty$. This proves the contrapositive of the converse.

(b) We claim μ is σ-finite $\iff$ for each $x \in \mathbf{R}$, there are only finitely many $n \in \mathbf{Z}^+$ for which $x_n = x$.

($\Rightarrow$): We show the contrapositive. Suppose $(\exists x \in \mathbf{R})\#\{n : x_n = x\} = +\infty$. Then for any sequence $\{A_m : m = 1, 2, \dots\} \subset \mathcal{B}(\mathbf{R})$,

$$\mathbf{R} = \cup_m A_m \;\Rightarrow\; (\exists m) x \in A_m \;\Rightarrow\; \mu(A_m) = \infty.$$

Hence μ cannot be σ-finite.

($\Leftarrow$): Write $\mathbf{R} = \bigcup_{i=0}^{\infty} A_i$ where $A_i = \{x_i\}$ is a single-element set for $i = 1, 2, \dots$, and $A_0 = \mathbf{R} \setminus \{x_n : n = 1, 2, \dots\}$. Evidently $A_i \in \mathcal{B}(\mathbf{R})$ for all i, $\mu(A_0) = 0$, and by hypothesis $\mu(A_i) < +\infty$ for every $i \in \mathbf{Z}^+$. $\square$

7. Show that a subset $B \subset \mathbf{R}$ is Lebesgue measurable if and only if

$$\lambda^*(I) = \lambda^*(I \cap B) + \lambda^*(I \cap B^c)$$

for every open interval $I \subset \mathbf{R}$.

Solution: Recall the definition that B is Lebesgue measurable (namely λ^*-measurable) iff $(\forall S \subset \mathbf{R})\ \lambda^*(S) = \lambda^*(B \cap S) + \lambda^*(B^c \cap S)$. We now show that

$$(\forall S \subset \mathbf{R})\lambda^*(S) = \lambda^*(B \cap S) + \lambda^*(B^c \cap S) \iff (\forall \text{ intervals } I)\lambda^*(I) = \lambda^*(B \cap I) + \lambda^*(B^c \cap I).$$

($\Rightarrow$): True for all $S \subset \mathbf{R} \Rightarrow$ true for intervals $S = I$.

($\Leftarrow$): Suppose $S \subset \mathbf{R}$ is given. Subadditivity implies $\lambda^*(S) \leq \lambda^*(B \cap S) + \lambda^*(B^c \cap S)$, so it suffices to prove the reverse inequality

$$\lambda^*(S) \geq \lambda^*(B \cap S) + \lambda^*(B^c \cap S).$$

This holds trivially if $\lambda^*(S) = +\infty$, so we may assume WOLOG that $\lambda^*(S)$ is finite.

By the approximation property of inf, for any $\epsilon > 0$ we can find a sequence of intervals $\{I_n : n = 1, 2, \ldots\}$ satisfying $S \subset \cup_n I_n$ and $\sum_n \lambda(I_n) < \lambda^*(S) + \epsilon$. Then

$$\begin{aligned}
\lambda^*(S) &> -\epsilon + \sum_n \lambda(I_n) \;=\; -\epsilon + \sum_n \lambda^*(I_n) \qquad \text{since } I_n \text{ is measurable,} \\
&= -\epsilon + \sum_n [\lambda^*(B \cap I_n) + \lambda^*(B^c \cap I_n)] \qquad \text{by hypothesis,} \\
&\geq -\epsilon + \lambda^*(\cup_n [B \cap I_n]) + \lambda^*(\cup_n [B^c \cap I_n]) \qquad \text{by subadditivity of } \lambda^*, \\
&\geq -\epsilon + \lambda^*(B \cap S) + \lambda^*(B^c \cap S) \qquad \text{by monotonicity of } \lambda^*.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, conclude that the desired reverse inequality holds. □

8. Let $(X, \mathcal{A}, \mu)$ be a measure space and let μ^* denote the outer measure defined by μ using $\mathcal{A}$, namely

$$\mu^*(S) = \inf\{\mu(A) : S \subset A, A \in \mathcal{A}\}$$

Define the *inner measure* $\mu_* : 2^X \to [0, +\infty]$ by

$$\mu_*(S) = \sup\{\mu(A) : A \subset S, A \in \mathcal{A}\}$$

(a) Prove that $\mu_*(S) \leq \mu^*(S)$ for every $S \subset X$.

(b) Prove that for any subset $S \subset X$ (not necessarily in $\mathcal{A}$) there are sets $A_0, A_1 \in \mathcal{A}$ satisfying $A_0 \subset S \subset A_1$ and

$$\mu(A_0) = \mu_*(S); \qquad \mu^*(S) = \mu(A_1).$$

Solution: (a) If $\mu_*(S) = +\infty$, then $\mu^*(S) = +\infty$ by monotonicity.

Else $\mu_*(S) < +\infty \Rightarrow (\forall \epsilon > 0)(\exists E \subset S, E \in \mathcal{A})\ \mu_*(S) < \mu(E) + \epsilon = \mu^*(E) + \epsilon$. But by monotonicity, $\mu^*(E) \leq \mu^*(S)$, so

$$(\forall \epsilon > 0)\ \mu_*(S) < \mu^*(S) + \epsilon.$$

Conclude that $\mu_*(S) \leq \mu^*(S)$.

(b) There are three cases to consider.

If $\mu^*(S) = \infty$, then we may take $A_1 = X$, for then $S \subset X = A_1$ implies $\mu(A_1) = \infty = \mu^*(S)$.

If $\mu_*(S) = \infty$, then there must be an ascending chain $\{E_n\} \subset \mathcal{A}$ of measurable subsets of S satisfying $(\forall n)\ \mu(E_n) \geq n$. Put $A_0 = \cup_n E_n \in \mathcal{A}$; then $A_0 \subset S$ and $\mu(A_0) = \lim_{n\to\infty} \mu(E_n) = +\infty$ by continuity of the measure μ.

It remains to consider the case $\mu_*(S) \leq \mu^*(S) < \infty$. Let $\{E_n\} \subset \mathcal{A}$ be an ascending chain of measurable subsets of S satisfying

$$\mu(E_n) \leq \mu_*(S) \leq \mu(E_n) + \frac{1}{n}.$$

Put $A_0 = \cup_n E_n \in \mathcal{A}$; then $A_0 \subset S$ and $\mu(A_0) = \lim_{n\to\infty} \mu(E_n) = \mu_*(S)$ by continuity of the measure μ.

Likewise, let $\{F_n\} \subset \mathcal{A}$ be a descending chain of measurable supersets of S satisfying

$$\mu(F_n) - \frac{1}{n} \leq \mu^*(S) \leq \mu(F_n).$$

Put $A_1 = \cap_n F_n \in \mathcal{A}$; then $S \subset A_1$ and $\mu(A_1) = \lim_{n\to\infty} \mu(F_n) = \mu^*(S)$, again by continuity of the measure μ. □

9. Say that the measure space $(X, \mathcal{A}, \mu)$ is *complete* iff whenever $S \subset X$ and there is some $A \in \mathcal{A}$ with $S \subset A$ and $\mu(A) = 0$, we may conclude that $S \in \mathcal{A}$.

   (a) Prove that if μ^* is an outer measure on X, and $\mathcal{A}$ is the σ-algebra of μ^*-measurable sets, then $(X, \mathcal{A}, \mu^*)$ is complete.

   (b) Decide whether $(\mathbf{R}, \mathcal{B}(\mathbf{R}), \lambda)$ is complete.

   Solution: (a) As proved in class, if μ^* is an outer measure, any set S with outer measure 0 is μ^*-measurable. But by monotonicity of outer measures, $S \subset A \in \mathcal{A}$ implies $0 \le \mu^*(S) \le \mu^*(A)$. Thus if $\mu^*(A) = 0$, we may conclude that $\mu^*(S) = 0$, so $S \in \mathcal{A}$.

   (b) $\mathcal{B}(\mathbf{R})$ is not complete with respect to Lebesgue measure λ.

   To prove this by counterexample, let $C \subset [0, 1] \subset \mathbf{R}$ be the Cantor set. Then C is a countable union of intervals, so $C \in \mathcal{B}(\mathbf{R})$. Also, $\lambda(C) = 0$, so any subset of C is Lebesgue measurable with measure 0.

   However, there is a one-to-one correspondence $f : [0, 1] \to C$ defined by

   $$x = 0.b_1 b_2 \ldots \text{ (base 2) } \mapsto f(x) = 0.(2b_1)(2b_2) \ldots \text{ (base 3)}.$$

   This f is uniformly continuous, since $|x - y| < 2^{-k}$ implies that the first k binary places of x and y must agree, so the first k ternary places of $f(x)$ and $f(y)$ agree, so $|f(x) - f(y)| \le 3^{-k}$.

   We also know that there exists a subset $E \subset [0, 1]$ which is not Lebesgue measurable, hence not in $\mathcal{B}(\mathbf{R})$. But then $f(E) \subset C$ is Lebesgue measurable, but $f(E) \notin \mathcal{B}(\mathbf{R})$, because the preimage of a Borel set by a continuous function is a Borel set. □

10. Define the *completion* of the σ-algebra $\mathcal{A}$ under μ to be the collection $\mathcal{A}_\mu$ of subsets $S \subset X$ for which there exist $E, F \in \mathcal{A}$ satisfying $E \subset S \subset F$ and $\mu(F \setminus E) = 0$.

    (a) Prove that $\mathcal{A} \subset \mathcal{A}_\mu$.

    (b) Prove that $S \in \mathcal{A}_\mu$ if and only if $\mu_*(S) = \mu^*(S)$. (Hint: assume exercise 8.) (ERRATUM: Part of this result may be false if $\mu^*(S) = \mu_*(S) = \infty$.)

    (c) Prove that $\mathcal{A}_\mu$ is a σ-algebra. (ERRATUM: this may not be true unless μ is σ-finite.)

    Solution:

    (a) Given $S \in \mathcal{A}$, choose $E = F = S$.

    (b) ($\Rightarrow$): Suppose $S \in \mathcal{A}_\mu$ with $E, F \in \mathcal{A}$ satisfying $E \subset S \subset F$ and $\mu(F \setminus E) = 0$. Then

    $$\mu(E) \le \mu_*(S) \le \mu^*(S) \le \mu(F),$$

    by exercise 8a and the definitions of inner and outer measure as sup and inf, respectively. But $F = E \cup (F \setminus E)$ is a disjoint union, so $\mu(F) = \mu(E) + \mu(F \setminus E) = \mu(E)$. Conclude that $\mu_*(S) = \mu^*(S)$.

    ($\Leftarrow$): By exercise 8b, find $E, F \in \mathcal{A}$ with $E \subset S \subset F$ and $\mu(E) = \mu_*(S) = \mu^*(S) = \mu(F)$.

    Now $F = E \cup (F \setminus E)$ is a disjoint decomposition, so $\mu(F) = \mu(E) + \mu(F \setminus E)$.

    Since $\mu^*(S) = \mu_*(S) < \infty$, then $\mu(F \setminus E) = \mu(F) - \mu(E) = 0$ and we have shown $S \in \mathcal{A}_\mu$.

    (c) NO SOLUTION.

    □