Ma 4121: Introduction to Lebesgue Integration
Solutions to Homework Assignment 5

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Due Thursday, April 11th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. Verify that the trigonometric system $\{\frac{1}{\sqrt{\pi}}\sin nx : n = 1, 2, \ldots\} \cup \{\frac{1}{\sqrt{\pi}}\cos nx : n = 1, 2, \ldots\} \cup \{\frac{1}{\sqrt{2\pi}}\}$ is orthonormal in $L^2([0, 2\pi])$.

   Solution: Compute the integrals:
   (1) $\frac{1}{\sqrt{2\pi}}$ is orthogonal to $\frac{1}{\sqrt{\pi}}\sin nx$ and $\frac{1}{\sqrt{\pi}}\cos nx$ for every $n = 1, 2, \ldots$, since

   $$\int_0^{2\pi} 1 \cdot \sin nx\, dx = \int_0^{2\pi} 1 \cdot \cos nx\, dx = 0.$$

   It has norm 1 since $\int_0^{2\pi} \left(\frac{1}{\sqrt{2\pi}}\right)^2 dx = 1$.
   (2) $\frac{1}{\sqrt{\pi}}\sin nx$ is orthogonal to $\frac{1}{\sqrt{\pi}}\cos mx$ for every $n, m = 1, 2, \ldots$, since

   $$\int_0^{2\pi} \sin nx \cdot \cos mx\, dx = \frac{1}{2}\int_0^{2\pi} [\sin(n+m)x + \sin(n-m)x]\, dx = 0.$$

   (3) $\frac{1}{\sqrt{\pi}}\sin nx$ is orthogonal to $\frac{1}{\sqrt{\pi}}\sin mx$ for every $n, m = 1, 2, \ldots$ with $n \neq m$, since then $n - m \neq 0$ and $n + m \neq 0$ and thus

   $$\int_0^{2\pi} \sin nx \cdot \sin mx\, dx = \frac{1}{2}\int_0^{2\pi} [\cos(n-m)x - \cos(n+m)x]\, dx = 0.$$

   (4) $\frac{1}{\sqrt{\pi}}\cos nx$ is orthogonal to $\frac{1}{\sqrt{\pi}}\cos mx$ for every $n, m = 1, 2, \ldots$ with $n \neq m$, since then $n - m \neq 0$ and $n + m \neq 0$ and thus

   $$\int_0^{2\pi} \cos nx \cdot \cos mx\, dx = \frac{1}{2}\int_0^{2\pi} [\cos(n-m)x + \cos(n+m)x]\, dx = 0.$$

   (5) Observing that $\cos^2 nx + \sin^2 nx = 1$ for all x and all $n = 1, 2, \ldots$, and that $\cos^2 nx$ and $\sin^2 nx$ have the same integral on $[0, 2\pi]$ since they are each 2π-periodic and each is just a translate by $\pi/2$ of the other, we conclude that

   $$\int_0^{2\pi} \cos^2 nx\, dx = \int_0^{2\pi} \sin^2 nx\, dx = \frac{1}{2}\int_0^{2\pi} 1\, dx = \pi.$$

Hence the factor $\frac{1}{\sqrt{\pi}}$ makes them normal. $\square$

2. Let $\{\phi_0, \phi_1, \ldots\} \subset X$ be an orthonormal system of functions, where X is an inner product space in which $\|f\| = 0 \iff f = 0$. Prove that the following three statements are equivalent:

   (a) If $\langle f, \phi_n \rangle = \langle g, \phi_n \rangle$ for all $n = 0, 1, 2, \ldots$, then $f = g$.

   (b) If $\langle f, \phi_n \rangle = 0$ for all n, then $f = 0$.

   (c) If T is an orthonormal system such that $\{\phi_0, \phi_1, \ldots\} \subset T$, then $T = \{\phi_0, \phi_1, \ldots\}$.

   Solution: We prove that a⇒b⇒c⇒a.

   (a⇒b): Since $\langle 0, \phi_n \rangle = 0$ for all $n = 0, 1, 2, \ldots$, use (a) to conclude that $f = 0$.

   (b⇒c): Suppose $\psi \in T$ but $\psi \notin \{\phi_0, \phi_1, \ldots\}$. Then $\langle \psi, \phi_n \rangle = 0$ by orthogonality, so $\psi = 0$ by (b).

   (c⇒a): Suppose we have 2 functions f, g with $f \neq g$ but $\langle f, \phi_n \rangle = \langle g, \phi_n \rangle$ for all $n = 0, 1, 2, \ldots$. Then $h = f - g \neq 0$ is orthogonal to ϕ_n for all n and $\|h\| \neq 0$, so $T \stackrel{\text{def}}{=} \{h/\|h\|\} \cup \{\phi_0, \phi_1, \ldots\}$ is an orthonormal system which contains $\{\phi_0, \phi_1, \ldots\}$. By (c), $h = \phi_n$ for some n. But then $1 = \langle h, \phi_n \rangle = \langle f, \phi_n \rangle - \langle g, \phi_n \rangle \Rightarrow \langle f, \phi_n \rangle \neq \langle g, \phi_n \rangle$, contradicting our assumption about f and g. $\square$

For Problems 3 and 4, define $C([0,1])$ to be the space of complex-valued continuous functions on the compact interval $[0,1]$, with $\langle f, g \rangle \stackrel{\text{def}}{=} \int_0^1 f(t)\bar{g}(t)\,dt$ for $f, g \in C([0,1])$, and $\|f\| \stackrel{\text{def}}{=} \sqrt{\langle f, f \rangle}$.

3. For $f \in C([0,1])$, prove that $\|f\| = 0 \iff f = 0$.

   Solution: If $f \in C([0,1])$ and $f(t_0) \neq 0$ for some $t_0 \in [0,1]$, then $|f(t_0)|^2 \neq 0$ is a continuous, hence integrable function on $[0,1]$ which satisfies $(\exists \epsilon > 0)(\forall t \in B(t_0, \epsilon) \cap [0,1])\ |f(t)|^2 > \frac{1}{2}|f(t_0)|^2$. But then

$$\|f\|^2 \geq \int_{B(t_0,\epsilon)\cap[0,1])} |f(t)|^2\,dt \geq \left(\frac{\epsilon}{2}\right)\left(\frac{1}{2}|f(t_0)|^2\right) > 0.$$

   Thus $f \neq 0 \Rightarrow \|f\| \neq 0$, *i.e.*, $\|f\| = 0 \Rightarrow f = 0$. The converse $f = 0 \Rightarrow \|f\| = 0$ is trivial since $\|0\|^2 = \int_0^1 0^2\,dt = 0$. $\square$

4. Prove that the set $\{e^{2\pi int} : n \in \mathbf{Z}\} \subset C([0,1])$ is an orthonormal system satisfying all three conditions of Problem 2.

   Solution: The functions are continuous and are seen to be orthonormal by an argument similar to that in Problem 1. They belong to a space $C([0,1])$ which by Problem 3 satisfies the hypothesis in Problem 2. Hence it suffices to show that any one of the equivalent conditions (a), (b), or (c) of Problem 2 is satisfied by $\{e^{2\pi int} : n \in \mathbf{Z}\}$.

   We choose (b). Put $\phi_n(t) \stackrel{\text{def}}{=} e^{2\pi int}$ and suppose that $\langle f, \phi_n \rangle = 0$ for all $n \in \mathbf{Z}$. Then the (exponential) Fourier coefficients generated by f are all zeroes, and so the sine/cosine Fourier coefficients are also all zeroes. Thus the Fourier series generated by f converges trivially to 0 at each $t \in [0,1]$. By Fejér's theorem it converges to $f(t)$, so we conclude that $f = 0$. $\square$

5. Put $I = [0, 1] \subset \mathbf{R}$. Suppose that $\mathcal{H} = \{\psi_n : n = 0, 1, 2, \ldots\} \subset L^2(I)$ is the orthogonal system of Haar functions defined for $n = 0$ by $\psi_0 = \mathbf{1}_I$, and for $0 < n = 2^j + k$ with $0 \leq k < 2^j$ is defined by

$$\psi_n(x) = \begin{cases} 1, & \text{if } \frac{k}{2^j} \leq x < \frac{k+\frac{1}{2}}{2^j}, \\ -1, & \text{if } \frac{k+\frac{1}{2}}{2^j} \leq x < \frac{k+1}{2^j}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that if $f \in L^2(I)$ and $\langle f, \psi_n \rangle = 0$ for all $n = 0, 1, 2, \ldots$, then $f = 0$ *a.e.* on I.

Solution: As shown in class, for a bounded interval such as I we have $L^2(I) \subset L(I)$. Thus, given $f \in L^2(I)$, we may find a sequence of linear combinations of Haar functions, $\{h_n : n = 1, 2, \ldots\} \subset \operatorname{span} \mathcal{H}$, satisfying

$$\lim_{n\to\infty} \|f - h_n\| = 0.$$

With this sequence, given $\epsilon > 0$ we may find $N < \infty$ such that

$$n \geq N \Rightarrow \|f - h_n\| < \epsilon.$$

Now $(\forall n)\, \langle f, \psi_n \rangle = 0 \;\Rightarrow\; (\forall n)\, \langle f, h_n \rangle = 0$. In particular, choosing $n \geq N$ gives

$$\|f\|^2 = |\|f\|^2 - 0| = |\|f\|^2 - \langle f, h_n \rangle\,| = |\,\langle f, f - h_n \rangle\,| \leq \|f\|\|f - h_n\|.$$

where we have used the Cauchy-Schwarz inequality at the last step.

Hence, either $\|f\| = 0 < \epsilon$ or else we may divide by $\|f\|$ to get $\|f\| \leq \|f - h_n\| < \epsilon$. Since $\epsilon > 0$ was arbitrary, conclude that $\|f\| = 0$.

Finally, note that $\|f\| = 0$ iff $f = 0$ *a.e.* on I. □

6. Show that $x = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$, if $0 < x < 2\pi$.

Solution: The Fourier coefficients of the 2π-periodic function that agrees with $f(x) = x$ on $(0, 2\pi)$ are evaluated as follows:

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} x \cos nx\, dx = 0, \quad \text{if } n > 0; \qquad a_0 = \frac{1}{\pi}\int_0^{2\pi} x\, dx = 2\pi; \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} x \sin nx\, dx = -\frac{2}{n}, \quad \text{if } n > 0. \end{aligned}$$

Thus $x = f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

Since $f(x) = x$ has bounded variation on every compact interval, Jordan's test (Theorem 11.16) implies that the Fourier series for $f(x)$ converges to $[f(x+) + f(x-)]/2$ at each $x \in (0, 2\pi)$. But this is $f(x)$, since f is continuous at each of those points. □

7. Show that $\dfrac{x^2}{2} = \pi x + 2\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^2} - 2\sum_{n=1}^{\infty} \dfrac{1}{n^2}$, if $0 \le x \le 2\pi$. Conclude that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$. (Hint: integrate the formula from Problem 6.)

Solution: By Theorem 11.6c, the Fourier series in Solution 6 can be integrated term by term and the integrated series converges uniformly on every interval. Thus, for $0 \le x \le 2\pi$,

$$\frac{x^2}{2} = \int_0^x t\,dt = \pi x - 2\sum_{n=1}^{\infty} \int_0^x \frac{\sin nt}{n}\,dt = \pi x + 2\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Now put $x = \pi$. Then $\cos nx = (-1)^n$ and by evaluating both sides we get the identity

$$\frac{\pi^2}{2} = \pi^2 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \frac{\pi^2}{2} = 4\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}.$$

But since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges absolutely, we can rearrange the summation into the odd and even parts. The even part is just $1/4$ of the original series, and we have

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{k=1}^{\infty} \frac{1}{(2k)^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{3}\sum_{n=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{6}.$$

Putting this into the last sum in the Fourier series gives the result. □

For Problems 8 and 9, define a 2π-periodic function f as follows:

$$f(t) = \begin{cases} 1, & \text{if } 0 < t < \pi; \\ -1, & \text{if } -\pi < t < 0; \\ 0, & \text{if } t = -\pi,\ t = 0, \text{ or } t = \pi. \end{cases}$$

8. Show that $f(x) = \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{\sin(2n-1)x}{2n-1}$ for every $x \in \mathbf{R}$.

Solution: Since this function is odd about $x = 0$, we can find its Fourier coefficients as follows:

$$a_0 = a_n = 0; \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\left[\frac{-\cos nt}{n}\right]\Bigg|_0^{\pi} = \begin{cases} \frac{4}{\pi n}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

The function f has bounded variation on every subinterval, so Jordan's test implies that its Fourier series converges to $[f(x+) + f(x-)]/2$, which is the same as $f(x)$, at each $x \in \mathbf{R}$. □

9. Let $s_n(x)$ be the partial sum of the first n terms of the Fourier series of the function f defined above. Show that for any $\epsilon > 0$,

$$\lim_{n\to\infty}\left[\max_{|x|<\epsilon} s_n(x) - \min_{|x|<\epsilon} s_n(x)\right] = \frac{4}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt.$$

(Hint: see problem 11.19 on pp.338–339 of the text.)

This result is known as *Gibbs' phenomenon.*

Solution: First we represent the partial sum as an integral and use Equation 15 on p.196 of the text:

$$s_n(x) \overset{\text{def}}{=} \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1} = \frac{4}{\pi}\sum_{k=1}^{n}\int_0^x \cos(2k-1)t\,dt = \frac{2}{\pi}\int_0^x \frac{\sin 2nt}{\sin t}\,dt.$$

This partial sum is a differentiable function, so its maxima and minima lie at critical points. Using the Fundamental Theorem of Calculus, we differentiate the integral and compute $s_n'(x) = (2\sin 2nx)/(\pi \sin x)$. For $0 < x < \pi$, this vanishes on the set $\{x_m = \frac{m\pi}{2n} : m = 1, 2, \ldots, 2n-1\}$. Now x_m will be a local maximum at odd m and a local minimum at even m; this is easily seen from checking the sign of $s_n'(x)$ on the intervals between adjacent x_m. Using the fact that $\sin t$ increases on $[0, \frac{\pi}{2}]$, we can conclude that $\{s_n(x_{2m})\} \uparrow$ and $\{s_n(x_{2m-1})\} \downarrow$ as m increases; thus, the maximum of $s_n(x)$ on $(0, \frac{\pi}{2})$ is attained at $x_1 = \frac{\pi}{2n}$. From the same fact, we conclude that $s_n(x) > 0$ for $0 < x < \frac{\pi}{2}$.

Since s_n' is an even function, s_n is an odd function, so that $-x_m$ will be a local minimum at odd m and a local maximum at even m. Thus, the minimum of $s_n(x)$ on $(-\frac{\pi}{2}, 0)$ is attained at $-\frac{\pi}{2n}$, and it is $-s_n(\frac{\pi}{2n})$. This negative number is smaller than any of the positive values taken by $s_n(x)$ on $x \in (0, \frac{\pi}{2})$, so $-s_n(\frac{\pi}{2n})$ is the minimum of $s_n(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Similarly, $s_n(\frac{\pi}{2n})$ is the maximum of $s_n(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Hence, as soon as $n > 0$ is so large that $\frac{\pi}{2n} < \epsilon < \frac{\pi}{2}$, we will have

$$\max_{|x|<\epsilon} s_n(x) - \min_{|x|<\epsilon} s_n(x) = 2s_n(\frac{\pi}{2n}) = \frac{4}{\pi}\int_0^{\frac{\pi}{2n}} \frac{\sin 2nt}{\sin t}\,dt = \frac{4}{2n\pi}\int_0^{\pi}\frac{\sin u}{\sin\frac{u}{2n}}\,du = \frac{4}{\pi}\int_0^{\pi}\left(\frac{\frac{u}{2n}}{\sin\frac{u}{2n}}\right)\frac{\sin u}{u}\,du.$$

We have substituted $t \leftarrow \frac{1}{2n}u$ to get the last expression. If we define $v/\sin v = 1$ at $v = 0$, then the (continuous) functions in parentheses tends uniformly to 1 on the compact interval $[0, \pi]$ as $n \to \infty$. Furthermore, $\frac{\sin u}{u}$ is continuous and bounded on $(0, \pi]$, hence is Lebesgue integrable on $[0, \pi]$. Hence by the Bounded Convergence Theorem, we have

$$\lim_{n\to\infty}\left[\max_{|x|<\epsilon} s_n(x) - \min_{|x|<\epsilon} s_n(x)\right] = \frac{4}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt.$$

□

10. Prove that if $f \in L([0, 2\pi])$ and $f'(x_0)$ exists at some point $x_0 \in (0, 2\pi)$, then the Fourier series generated by f converges at x_0.

Solution: We may assume without loss that $f(x_0) = 0$, since if we prove the result for $f_0(x) \overset{\text{def}}{=} f(x) - f(x_0)$ we will get the Fourier series for f by adding the constant $f(x_0)$ to the Fourier series for f_0.

We now define a 2π-periodic function h by specifying its values on $[0, 2\pi]$ as follows:

$$h(t) = f(t + x_0), \quad \text{if } 0 < t < 2\pi; \qquad h(0) = h(2\pi) = 0.$$

Then there is some 2π-periodic function $g = g(t)$ such that $h(t) = \left(e^{-it} - 1\right) g(t)$. Note that the quotient function g is Lebesgue integrable, since $h(t)/\left(e^{-it} - 1\right)$ has a finite limit as $t \to 0$. But then the exponential Fourier coefficients for h and g are related by $\hat{h}(k) = \hat{g}(k+1) - \hat{g}(k)$, so that we obtain a telescoping Fourier series for $h(0)$:

$$\sum_{k=-n}^{m-1}\hat{h}(k) = \hat{g}(m) - \hat{g}(-n).$$

By the Riemann-Lebesgue lemma, this series converges to zero as $n, m \to \infty$ in any manner. To complete the proof, note that the Fourier series for $h(0)$ is just the Fourier series for $f(x_0)$.
$\square$