Ma 4121: Introduction to Lebesgue Integration
Solutions to Homework Assignment 6

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Due Thursday, April 25th, 2013

Please return your solutions to the instructor by the end of class on the due date. You may collaborate on these problems but you must write up your own solutions. **Late homework will not be accepted.**

1. For fixed $c \in (0,1)$, define $f : \mathbf{R}^2 \to \mathbf{R}$ as follows:

$$f(x,y) \stackrel{\text{def}}{=} \begin{cases} (1-y)^c/(x-y)^c, & \text{if } 0 < x < 1 \text{ and } 0 \le y < x; \\ 0, & \text{otherwise.} \end{cases}$$

Prove that $f \in L(\mathbf{R}^2)$ and evaluate $\int_{\mathbf{R}^2} f$.

Solution: In fact, f is an upper function. For $\epsilon > 0$, define

$$f_\epsilon(x,y) \stackrel{\text{def}}{=} \begin{cases} (1-y)^c/(x-y)^c, & \text{if } 0 < x < 1 \text{ and } 0 \le y < x-\epsilon; \\ 0, & \text{otherwise.} \end{cases}$$

Then $f_\epsilon \ge 0$ and $f_\epsilon \nearrow f$ as $\epsilon \to 0$. Also, f_ϵ is bounded and vanishes outside a bounded triangular region, so $f_\epsilon \in L(\mathbf{R}^2)$ for each $\epsilon > 0$.

Finally, use Fubini's theorem to calculate

$$\begin{aligned} \int_{\mathbf{R}^2} f_\epsilon &= \int_{y=0}^{1-\epsilon} \int_{x=y+\epsilon}^{1} \frac{(1-y)^c}{(x-y)^c} = \int_{y=0}^{1-\epsilon} \left[\frac{(1-y)^c (x-y)^{1-c}}{1-c} \right]_{x=y+\epsilon}^{1} \\ &= \int_{y=0}^{1-\epsilon} \frac{(1-y)^c [(1-y)^{1-c} - \epsilon^{1-c}]}{1-c} = \int_{y=0}^{1-\epsilon} \frac{1-y}{1-c} - \epsilon^{1-c} \int_{y=0}^{1-\epsilon} \frac{(1-y)^c}{1-c} \\ &= \left[\frac{y - y^2/2}{1-c} \right]_{y=0}^{1-\epsilon} - \epsilon^{1-c} \int_{y=0}^{1-\epsilon} \frac{(1-y)^c}{1-c} = (1-\epsilon)(\frac{1}{2} + \frac{\epsilon}{2}) - \epsilon^{1-c} \int_{y=0}^{1-\epsilon} \frac{(1-y)^c}{1-c}. \end{aligned}$$

This evidently converges to $1/2$ as $\epsilon \to 0$, showing that $f \in L(\mathbf{R}^2)$ with $\int f = 1/2$.

Note that integration in x first, then in y, gives a simple antiderivative. To find the limits of integration, it is useful to draw a graph of the triangular region. □

2. Suppose that $S \subset \mathbf{R}^2$ is a measurable set with the property that $\lambda(S_y) = 0$ for almost every $y \in \mathbf{R}$, where λ is 1-dimensional Lebesgue measure on $\mathbf{R}$, and

$$S_y \stackrel{\text{def}}{=} \{x \in \mathbf{R} : (x,y) \in S\}.$$

Prove that the 2-dimensional Lebesgue measure of S is zero. (Note: This is a partial converse to Theorem 15.5 on p.412 of our text.)

Solution: Put $f(x,y) \overset{\text{def}}{=} \mathbf{1}_S(x,y)$. Then $f \in M(\mathbf{R}^2)$. It suffices to show that $\int_{\mathbf{R}^2} f = 0$.

Suppose first that S is bounded. Then $f \in L(\mathbf{R}^2)$ and we may use Fubini's theorem to evaluate

$$\int_{\mathbf{R}^2} f = \int_{y\in\mathbf{R}} G(y) = 0, \qquad \text{since } G(y) = \int_{x\in\mathbf{R}} f(x,y) = \lambda(S_y) = 0, \quad a.e. y \in \mathbf{R}.$$

Now write $S = \bigcup_{n=1}^{\infty} S_n$, where $S_n = S \cap [-n,n]^2$. Then for each n, S_n is bounded and satisfies the same hypotheses as S, so it has 2-dimensional Lebesgue measure 0. Thus the 2-dimensional Lebesgue measure of S is at most the sum of countably many 0s, hence is zero. □

3. Suppose that $f_i : \mathbf{R} \to \mathbf{R}$ is defined and bounded on the compact interval $[a_i, b_i] \subset \mathbf{R}$. If $f_i \in L([a_i, b_i])$ for $i = 1, \ldots, n$, prove that

$$\int_Q f_1(x_1) \cdots f_n(x_n)\, d(x_1, \ldots, x_n) = \left(\int_{a_1}^{b_1} f_1(x_1)\, dx_1\right) \cdots \left(\int_{a_n}^{b_n} f_n(x_n)\, dx_n\right),$$

where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbf{R}^n$.

Solution: We prove this by induction on n. It is evidently true when $n = 1$, for then both sides are the same.

Suppose the result holds for $n-1$; let $g : \mathbf{R}^{n-1} \to \mathbf{R}$ be defined by $g(x_1, \ldots, x_{n-1}) = f(x_1) \cdots f(x_{n-1})$. This g is defined and bounded on $Q_{n-1} \overset{\text{def}}{=} [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}]$. For any $f_n \in L([a_n, b_n])$, the function $gf_n = g(x_1, \ldots, x_{n-1}) f_n(x_n)$ belongs to $L(Q)$ by Lebesgue's dominated convergence theorem and we have

$$\int_Q g(x_1, \ldots, x_{n-1}) f_n(x_n)\, d(x_1, \ldots, x_n) = \left(\int_{Q_{n-1}} g(x_1, \ldots, x_{n-1})\, d(x_1, \ldots, x_{n-1})\right) \left[\int_{a_n}^{b_n} f_n(x_n)\, dx_n\right],$$

by Fubini's theorem. Here $g(x_1, \ldots, x_{n-1})$ may be removed from the integral in x_n since it has no x_n-dependence, and the remaining function f_n is Lebesgue integrable by hypothesis. The result for n now follows from the inductive hypothesis. □

4. (a) Prove that $\int_{\mathbf{R}^2} e^{-x^2-y^2} = \pi$ by transforming the integral to polar coordinates

   (b) Use part(a) to prove that $\int_{\mathbf{R}} e^{-x^2} = \sqrt{\pi}$.

   (c) Use part (b) to prove that $\int_{\mathbf{R}^n} e^{-\|x\|^2} = \pi^{n/2}$.

   (d) Evaluate $\int_{\mathbf{R}} e^{-tx^2}$ for $t > 0$, and find t for which the value is 1.

Solution: (a) $\mathbf{R}^2 = \{(x,y) = (r\cos\theta, r\sin\theta) : 0 < r < \infty, 0 \le \theta < 2\pi\} \cup \{(0,0)\}$, and this mapping has Jacobian

$$J = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

with determinant $|J| = |r| = r$. Hence, by eq.30 on p.429 of our text,

$$\int_{\mathbf{R}^2} e^{-x^2-y^2} = \int_{\mathbf{R}^2} e^{-r^2} |J|,$$

which we may evaluate by Fubini's theorem as

$$\int_{\mathbf{R}^2} e^{-x^2-y^2}\, d(x,y) = \int_{\theta=0}^{2\pi} \left[\int_{r=0}^{\infty} e^{-r^2} r\, dr\right] d\theta = 2\pi \left[\frac{1}{2} \int_{u=0}^{\infty} e^{-u}\, du\right] = \pi,$$

after substituting $r^2 \leftarrow u$ and $r\,dr \leftarrow \frac{1}{2}du$.

(b) Since $e^{-x^2-y^2} = e^{-x^2}e^{-y^2}$, by using the results of exercise 3 above we get

$$\int_{\mathbf{R}^2} e^{-x^2-y^2} = \left(\int_{x\in\mathbf{R}} e^{-x^2}\right)\left(\int_{y\in\mathbf{R}} e^{-y^2}\right) = \left(\int_{x\in\mathbf{R}} e^{-x^2}\right)^2,$$

since the two integrals are the same except for the name of the variable. Combine with part (a) and the observation that the integrals must be positive to conclude that $\int_{\mathbf{R}} e^{-x^2} = \sqrt{\pi}$.

(c) Another application of exercise 3 above gives

$$\int_{x\in\mathbf{R}^n} e^{-\|x\|^2} = \left(\int_{x\in\mathbf{R}} e^{-x^2}\right)^n = \pi^{n/2},$$

where the last equality follows from part (b).

(d) Write $f(x) = e^{-tx^2}$ for $t > 0$, and put $g(x) \stackrel{\text{def}}{=} f(x/\sqrt{t}) = e^{-x^2}$. Thus, using part (b) above and the equation on p.407 of our text gives

$$\sqrt{\pi} = \int_{x\in\mathbf{R}} e^{-x^2} = \int_{x\in\mathbf{R}} g(x) = (\sqrt{t})^1 \int_{x\in\mathbf{R}} f(x) = \sqrt{t}\int_{x\in\mathbf{R}} e^{-tx^2}.$$

Thus $\int_{x\in\mathbf{R}} e^{-tx^2} = \sqrt{\pi}/\sqrt{t}$, and choosing $t = \pi$ gives

$$\int_{x\in\mathbf{R}} e^{-\pi x^2} = 1.$$

□

5. Let $V_n(a)$ denote the volume of the ball of radius a in $\mathbf{R}^n$, that is, the n-dimensional Lebesgue measure of the open set $\{x \in \mathbf{R}^n : \|x\| < a\}$.

(a) Prove that $V_n(a) = a^n V_n(1)$.

(b) Prove that, for $n \geq 3$, we have the formula

$$V_n(1) \;=\; V_{n-2}(1) \times \int_0^{2\pi}\left[\int_0^1 (1-r^2)^{n/2-1} r\,dr\right]d\theta \;=\; V_{n-2}(1)\frac{2\pi}{n}.$$

(c) Use the recursion in part (b) to conclude that

$$V_n(1) = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)},$$

where Γ is the special function defined on p.277 of our text.

Solution: In the following, write $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $x \in \mathbf{R}^n$, and let

$$B = B^n = B^n(0;1) = \{x \in \mathbf{R}^n : \|x\| < 1\} \subset \mathbf{R}^n$$

denote the unit open n-ball centered at 0.

(a) Observe that

$$V_n(1) = \int_{x\in\mathbf{R}^n} \mathbf{1}_B(x) \qquad \text{and} \qquad V_n(a) = \int_{x\in\mathbf{R}^n} \mathbf{1}_B(x/a),$$

so $V_n(a) = a^n V_n(1)$ by the relation on p.407 of our textbook.

(b) Write

$$\begin{aligned} B^n(0;1) &= \{x \in \mathbf{R}^n : \|x\|^2 < 1\} = \{(y,z) \in \mathbf{R}^n : y \in \mathbf{R}^{n-2}, z \in \mathbf{R}^2, \|y\|^2 + \|z\|^2 < 1\} \\ &= \{(y,z) \in \mathbf{R}^n : y \in \mathbf{R}^2, \|y\|^2 < 1; z \in \mathbf{R}^{n-2}, \|z\|^2 < 1 - \|y\|^2\} \\ &= \{(y,z) \in \mathbf{R}^n : y \in B^2(0;1), z \in B^{n-2}(0; 1 - \|y\|^2)\}. \end{aligned}$$

This suggests a method to evaluate $V_n(1)$ by iterated integration:

$$V_n(1) = \int_{y \in B^2(0;1)} \int_{z \in B^{n-2}(0;1-\|y\|^2)} 1 = \int_{y \in B^2(0;1)} V_{n-2}(\sqrt{1 - \|y\|^2}),$$

so using the scaling relation in part (a), we get

$$V_n(1) = \int_{y \in B^2(0;1)} (1 - \|y\|^2)^{(n-2)/2} V_{n-2}(1) = V_{n-2}(1) \int_{y \in B^2(0;1)} (1 - \|y\|^2)^{n/2-1}.$$

Evaluating the integral in polar coordinates gives

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} (1 - r^2)^{n/2-1} r \, dr d\theta = 2\pi \frac{1}{2} \int_{u=0}^{1} (1-u)^{n/2-1} \, du = \frac{2\pi}{n},$$

using elementary methods and the substitution $r \leftarrow \sqrt{u}$. Thus

$$V_n(1) = \frac{2\pi}{n} V_{n-2}(1).$$

(c) Prove this by induction on n.

First recall that $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Also, $\Gamma(1+x) = x\Gamma(x)$ for all positive real x, and thus $\Gamma(n+1) = n!$ for all nonnegative integers n. (Many references list these special values and relations, or else they may be derived by elementary methods.)

Check the case $n = 1$ by noting that

$$V_1(1) = 2 = \frac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} = \frac{\pi^{1/2}}{\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{\pi^{1/2}}{\Gamma(\frac{1}{2} + 1)}$$

Next, check the case $n = 2$ by observing that

$$V_2(1) = \pi = \frac{\pi^{2/2}}{1\Gamma(1)} = \frac{\pi^{2/2}}{\Gamma(2)} = \frac{\pi^{2/2}}{\Gamma(\frac{2}{2} + 1)}.$$

Now suppose that $n > 2$ and that the equation holds for all $k = 1, 2, \ldots, n-1$. Then by (b),

$$V_n(1) = \frac{2\pi}{n} V_{n-2}(1) = \frac{(2\pi)\pi^{(n-2)/2}}{n\Gamma(\frac{n-2}{2} + 1)} = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)},$$

proving the inductive step. □

6. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$f(x,y) = \begin{cases} e^y \sin x, & \text{if } x \text{ is rational;} \\ e^{-x^2-y^2}, & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $f \in L(\mathbf{R}^2)$ and compute $\int_{\mathbf{R}^2} f$.

Solution: Use the Tonelli-Hobson test, Th.15.8 on p.415 of our text. Since $\mathbf{Q}$ has 1-dimensional measure zero, the set $\mathbf{Q} \times \mathbf{R}$ has 2-dimensional measure zero by exercise 2 above, so f agrees almost everywhere with the function $g(x, y) = e^{-x^2-y^2}$.

Now $g > 0$ on $\mathbf{R}^2$, so $|g| = g$, and the iterated integral exists:

$$\int_{y\in\mathbf{R}} \left[\int_{x\in\mathbf{R}} |g(x, y)|\right] = \int_{y\in\mathbf{R}} \sqrt{\pi}\, e^{-y^2} = \pi,$$

using the results of exercise 4 above. Conclude that $g \in L(\mathbf{R}^2)$, so $f \in L(\mathbf{R}^2)$, with $\int f = \int g = \int |g| = \pi$. $\square$

7. Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ for $0 \le x \le 1$, $0 < y \le 1$, and put $f(0, 0) = 0$. Prove that both iterated integrals

$$\int_{y=0}^{1} \left[\int_{x=0}^{1} f(x, y)\, dx\right] dy, \quad \text{and} \quad \int_{x=0}^{1} \left[\int_{y=0}^{1} f(x, y)\, dy\right] dx$$

exist but are not equal. Conclude that $f \notin L([0, 1] \times [0, 1])$.

Solution: First note that for $(x, y) \neq (0, 0)$,

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right) = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial x}\left(\frac{-x}{x^2 + y^2}\right).$$

We may thus compute the iterated integrals by antidifferentiation:

$$\int_{x=0}^{1} \left(\int_{y=0}^{1} \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dy\right) dx = \int_{x=0}^{1} \left[\frac{y}{x^2 + y^2}\right]_{y=0}^{1} dx = \int_{x=0}^{1} \frac{1}{1 + x^2}\, dx \; > \; 0,$$

while

$$\int_{y=0}^{1} \left(\int_{x=0}^{1} \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx\right) dy = \int_{y=0}^{1} \left[\frac{-x}{x^2 + y^2}\right]_{y=0}^{1} dx = \int_{y=0}^{1} \frac{-1}{1 + y^2}\, dy \; < \; 0,$$

where both integrals exist because the integrands are continuous and bounded. $\square$

8. Let $f(x, y) = e^{-xy} \sin x \sin y$ for $x \ge 0$ and $y \ge 0$, and let $f(x, y) = 0$ otherwise. Prove that both iterated integrals

$$\int_{y\in\mathbf{R}} \left[\int_{x\in\mathbf{R}} f(x, y)\, dx\right] dy, \quad \text{and} \quad \int_{x\in\mathbf{R}} \left[\int_{y\in\mathbf{R}} f(x, y)\, dy\right] dx$$

exist and are equal, but that $f \notin L(\mathbf{R}^2)$. Explain why this does not contradict the Tonelli-Hobson test (theorem 15.8, p.415).

Solution: If one of the iterated integrals exists, then so will the other, and they will be equal, for the function satisfies $f(x, y) = f(y, x)$ for all $(x, y) \in \mathbf{R}^2$.

In Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products*, 5th edition, equation 3.893(1) on p.512 implies

$$\int_{y=0}^{\infty} e^{-xy} \sin y\, dy = \frac{1}{1 + x^2}, \qquad \text{if } x > 0,$$

so that

$$\int_{y=0}^{\infty} \left(\int_{x=0}^{\infty} e^{-xy} \sin x \sin y\, dx\right) dy = \int_{x=0}^{\infty} \frac{\sin x}{1 + x^2}\, dx < \infty,$$

by Lebesgue's dominated convergence theorem and comparison with $1/(1+x^2) \in L(\mathbf{R})$.

Now suppose toward contradiction that $|f| \in L(\mathbf{R}^2)$. Let

$$B \stackrel{\text{def}}{=} \{t \in \mathbf{R}^+ : |\sin t| \geq \frac{1}{\sqrt{2}}\} = \bigcup_{k=0}^{\infty} B_k,$$

where $B_k \stackrel{\text{def}}{=} [(k+\frac{1}{4})\pi, (k+\frac{3}{4})\pi]$. Then for $(x,y) \in B \times B \subset \mathbf{R}^2$, we have $|\sin x \sin y| \geq \frac{1}{2}$. We define a function $s : \mathbf{R}^2 \to \mathbf{R}$ as follows:

$$s(x,y) = \begin{cases} \min\{|f(s,t)| : (s,t) \in B_m \times B_n\}, & \text{if } (x,y) \in B_m \times B_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $s \in M(\mathbf{R}^2)$ since it is a countable sum of step functions, and $0 \leq s \leq |f|$ on $\mathbf{R}^2$. Thus $|f| \in L(\mathbf{R}^2)$ would imply that $s \in L(\mathbf{R}^2)$. But in fact

$$s(x,y) = \frac{1}{2} e^{-(m+\frac{3}{4})(n+\frac{3}{4})\pi^2}, \qquad \text{if } (x,y) \in B_m \times B_n,$$

and $|B_m \times B_n| = \pi^2/4$ for every $n, m = 0, 1, 2, \ldots$, so

$$\iint_{\mathbf{R}^2} s = \frac{\pi^2}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2} e^{-(\frac{3}{4}+m)(\frac{3}{4}+n)\pi^2} \geq \frac{\pi^2}{8} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-mn\pi^2}$$

Finally, we may use the integral test to show that the double series diverges, for

$$\iint_{x>0,y>0} e^{-xy\pi^2} = \frac{1}{\pi^2} \int_{x=0}^{\infty} \left(\int_{y=0}^{\infty} e^{-xy}\, dy \right) dx = \frac{1}{\pi^2} \int_{x=0}^{\infty} \frac{1}{x}\, dx = +\infty.$$

Hence $s \notin L(\mathbf{R}^2)$. Conclude that $|f| \notin L(\mathbf{R}^2)$, and thus that $f \notin L(\mathbf{R}^2)$.

This f is not a counterexample to the Tonelli-Hobson theorem precisely because $|f| \notin L(\mathbf{R}^2)$, and so neither of the iterated integrals for $|f|$ will be finite. □

9. Let $I = [0,1] \times [0,1]$, let $f(x,y) = (x-y)/(x+y)^3$ if $(x,y) \in I \setminus \{(0,0)\}$, and let $f(0,0) = 0$. Prove that $f \notin L(I)$ by considering the integrals

$$\int_{y=0}^{1} \left[\int_{x=0}^{1} f(x,y)\, dx \right] dy, \quad \text{and} \quad \int_{x=0}^{1} \left[\int_{y=0}^{1} f(x,y)\, dy \right] dx$$

Solution: From Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products*, 5th edition, we have

$$\int \frac{dx}{(y+x)^3} = \frac{1}{-2(y+x)^2}; \qquad \text{(2.111(1), p.68)}$$

and

$$\int \frac{x\, dx}{(y+x)^3} = -\left(x + \frac{y}{2}\right) \frac{1}{(y+x)^2}; \qquad \text{(2.114(2), p.69)}$$

Thus for $y > 0$, we have

$$\int_0^1 \frac{(x-y)}{(x+y)^3}\, dx = -\left[\frac{x+\frac{y}{2}}{(x+y)^2} \right]_{x=0}^{1} + \left[\frac{\frac{y}{2}}{(x+y)^2} \right]_{x=0}^{1} = \frac{-1}{(1+y)^2}.$$

Similary, by exchanging $x \leftrightarrow y$ and noting that $f(x,y) = -f(y,x)$ for all x, y, we get

$$\int_0^1 \frac{(x-y)}{(x+y)^3}\, dy = \frac{1}{(1+x)^2}, \qquad \text{for } x > 0.$$

Both of these functions have elementary antiderivatives, with which we may compute

$$\int_{y=0}^{1}\left[\int_{x=0}^{1} f\,dx\right]\,dy = \left[\frac{1}{1+y}\right]_0^1 = -\frac{1}{2}, \quad \text{while} \quad \int_{x=0}^{1}\left[\int_{y=0}^{1} f\,dy\right]\,dx = \left[\frac{-1}{1+x}\right]_0^1 = +\frac{1}{2}.$$

Since these are not equal, the original function f cannot belong to $L(I)$ or else there would be a contradiction with Fubini's theorem. □

10. Let $I = [0,1]\times[1,+\infty)$ and let $f(x,y) = e^{-xy} - 2e^{-2xy}$ if $(x,y) \in I$. Prove that $f \notin L(I)$ by considering the integrals

$$\int_{y=1}^{\infty}\left[\int_{x=0}^{1} f(x,y)\,dx\right]\,dy, \quad \text{and} \quad \int_{x=0}^{1}\left[\int_{y=1}^{\infty} f(x,y)\,dy\right]\,dx$$

Solution: Function f is continuous on I, hence it is measurable. It is also bounded, but I is not bounded. Were $f \in L(I)$, it would satisfy Fubini's theorem and we would have

$$\int_x\left(\int_y f\right) = \iint_I f = \int_y\left(\int_x f\right).$$

But

$$\int_x\left(\int_y f\right) = \int_{x=0}^{1}\left(\int_{y=1}^{\infty} f(x,y)\,dy\right)\,dx = \int_0^1 \frac{e^{-x}}{x}[e^{-x}-1]\,dx,$$

while

$$\int_y\left(\int_x f\right) = \int_{y=1}^{\infty}\left(\int_{x=0}^{1} f(x,y)\,dx\right)\,dy = \int_1^{\infty} \frac{e^{-y}}{y}[e^{-y}-1]\,dy = \int_0^1 \frac{e^{-1/z}}{z}[e^{-1/z}-1]\,dz,$$

as shown by elementary methods, with the last equality following from the substitution $y \leftarrow 1/z$.

But for all $z \in (0,1)$, the continuous and bounded integrand functions satisfy

$$\frac{1}{z}e^{-1/z}[e^{-1/z}-1] > \frac{1}{z}e^{-z}[e^{-z}-1],$$

so the two iterated integrals exist but cannot be equal. □