Ma 416: Complex Variables
Midterm Examination

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1. Find the real parts, imaginary parts, and absolute values of the complex numbers $z = (1-i)/(1+i)$ and $w = \exp(i\pi/3)$.

2. Find the domain of convergence of the following power series:

$$\text{(a)} \sum_{n=0}^{\infty} \frac{(z-2)^n}{n} \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

3. Suppose that an entire analytic function f has arbitrarily small periods. That is, suppose that there is an infinite sequence $\{p_k : k \in \mathbf{N}\}$ with $|p_k| \to 0$ as $k \to \infty$ such that $f(z + p_k) = f(z)$ for all k and all $z \in \mathbf{C}$. Prove that f must be constant.

4. Let $C = \{z : |z| = r\}$ be a circle of radius $r > 0$, centered at the origin in $\mathbf{C}$, equipped with the positive (counterclockwise) orientation. Let $n > 1$ be an integer. Compute $\int_C (1/z^n)\, dz$. (Hint: parametrize C.)

5. Let $D \subset \mathbf{C}$ be the closed disk of radius R centered at 0. Suppose that $f = f(z)$ is analytic on D and satisfies $|f(z)| \leq M$ for all $z \in D$. Prove that $|f'''(0)| \leq 6M/R^3$.

6. Classify the following singularities as removable, poles, or essential. If the singularity is a pole, state its order.

   (a) $(\sin z)/z$ at $z = 0$ (b) $\sin(1/z)$ at $z = 0$ (c) $1/(\sin z)^3$ at $z = 0$

7. Use the Residue Theorem to evaluate the improper integral $\int_{-\infty}^{\infty} (x^2+1)^{-1}\, dx$.