Ma 416: Complex Variables
Homework Assignment 4

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Due Thursday, September 29th, 2005

1. Let $f_n(x) = [x^n(1-x^n)]$ for $n = 1, 2, 3, \ldots$. Does the sequence $\{f_n(x)\}$ converge uniformly on $0 < x < 1$?

2. Use Cauchy's Inequalities to deduce Liouville's Theorem.

3. Let $D \subset \mathbf{C}$ be the closed diamond-shaped region with vertices $1, i, -1, -i$. Suppose that $f = f(z)$ is analytic on D and satisfies $|f(z)| \leq M$ for all $z \in D$. Prove that $|f'(0)| \leq M\sqrt{2}$ and $|f''(0)| \leq 4M$.

4. Suppose that $f(z)$ is analytic on $|z| < 2$. Define $F_0(z) = f(z)$ and $F_{n+1}(z) = \int_0^z F_n(w)\, dw$ for $n \geq 0$. Prove that if $\{F_n(z)\}$ converges uniformly on $|z| < 1$, then $f(z) = ce^z$ for some constant c.

5. Recall that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$ for all real x. Show that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z,$$

   for all complex z. (Hint: use the uniform convergence theorem and the coincidence principle.)

6. Compute $\Gamma(3/2)$ and $\Gamma(-1/2)$.